1 Lecture review

1.1 Lagrange Multipliers (General Theory)

1. Lagrange multipliers are used to solve constrained optimization problems. In these problems, there is a target function which we would like to maximize or minimize subject to some constraint.

2. (One constraint) If we wish to maximize or minimize \( f \) subject to a constraint \( g = 0 \), then we must solve the equation \( \nabla f = \lambda \nabla g \). The “\( \lambda \)” here is called the Lagrange multiplier.

3. (Two constraints) If we wish to maximize or minimize \( f \) subject to two constraints \( g = 0 \) and \( h = 0 \), then we must solve the equation \( \nabla f = \lambda \nabla g + \mu \nabla h \). The “\( \lambda \), \( \mu \)” here are called the Lagrange multipliers.

1.2 Lagrange Multipliers, 2-variable case

1. Suppose we want to maximize/minimize \( f(x, y) \) subject to the constraint \( g(x, y) = 0 \).

2. Geometrically, the constraint \( g(x, y) = 0 \) represents the equation of a curve in the \( xy \)-plane. So we are maximizing/minimizing the function \( f(x, y) \) on this curve.

3. The Lagrange multiplier equation is \( \nabla f = \lambda \nabla g \), which when written out in coordinates gives

\[
\begin{align*}
  f_x &= \lambda g_x, & f_y &= \lambda g_y, & g &= 0.
\end{align*}
\]

This gives three equations for the three variables \( x, y, \lambda \). Solving them and checking values will allow one to find the maxima and minima.

1.3 Lagrange Multipliers, 3-variable case, 1 constraint

1. Suppose we want to maximize/minimize \( f(x, y, z) \) subject to the constraint \( g(x, y, z) = 0 \).

2. Geometrically, the constraint \( g(x, y, z) = 0 \) represents the equation of a surface in space. So we are maximizing/minimizing the function \( f(x, y, z) \) on this surface.

3. The Lagrange multiplier equation is \( \nabla f = \lambda \nabla g \), which when written out in coordinates gives

\[
\begin{align*}
  f_x &= \lambda g_x, & f_y &= \lambda g_y, & f_z &= \lambda g_z, & g &= 0.
\end{align*}
\]

This gives four equations for the four variables \( x, y, z, \lambda \). Solving them and checking values will allow one to find the maxima and minima.

1.4 Lagrange Multipliers, 3-variable case, 2 constraints

1. Suppose we want to maximize/minimize \( f(x, y, z) \) subject to the constraints \( g(x, y, z) = 0 \) and \( h(x, y, z) = 0 \)

2. Geometrically, the constraints \( g(x, y, z) = 0 \) and \( h(x, y, z) = 0 \) together represent the equation of a curve in space. So we are maximizing/minimizing the function \( f(x, y, z) \) on this curve.

3. The Lagrange multiplier equation is \( \nabla f = \lambda \nabla g + \mu \nabla h \), which when written out in coordinates gives

\[
\begin{align*}
  f_x &= \lambda g_x + \mu h_x, & f_y &= \lambda g_y + \mu h_y, & f_z &= \lambda g_z + \mu h_z, & g &= 0, & h &= 0.
\end{align*}
\]

This gives five equations for the five variables \( x, y, z, \lambda, \mu \). Solving them and checking values will allow one to find the maxima and minima.
2 Problems

1. (2I-1) A rectangular box is placed in the first octant so that one corner $Q$ is at the origin and the three sides adjacent to $Q$ lie in the coordinate planes. The corner $P$ diagonally opposite $Q$ lies on the surface $f(x, y, z) = c$. Using Lagrange multipliers, tell for which point $P$ the box will have the largest volume, and tell how you know it gives a maximum point, if the surface is

(a) the plane $x + 2y + 3z = 18$  
(b) the ellipsoid $x^2 + 2y^2 + 4z^2 = 12$

2. (2F-1, 2I-2) Using Lagrange multipliers, find the point(s) on each of the following surfaces which is closest to the origin. (Hint: minimize the square of the distance.)

(a) $xyz^2 = 1$  
(b) $x^2 - yz = 1$  
(c) $x^3y^2z = 6\sqrt{3}$

3. (2I-3) A rectangular produce box is to be made of cardboard; the sides of single thickness, the ends of double thickness, and the bottom of triple thickness, with the top left open. Its volume is to be 1 cubic foot; what should be its proportions in order to use the least cardboard?

4. (2I-4) In an open-top wooden drawer, the two sides and back cost $2/sq. ft., the bottom $1/sq. ft. and the front $4/sq. ft. Using Lagrange multipliers, show that the following problems lead to the same set of three equations in $\lambda$, plus a different fourth equation, and they have the same solution.

(a) Find the dimensions of the drawer with largest capacity that can be made for a total wood cost of $72.

(b) Find the dimensions of the most economical drawer having volume 24 cu. ft.

5. (2F-5) A drawer in a chest has an open top; the bottom and back are made of cheap wood costing $1/sq. ft; the sides have to be thicker, and cost $2/sq. ft., while the front costs $4/sq. ft. for the better quality wood and finishing. The volume is to be 2.5 cu. ft. What dimensions will produce the drawer costing the least to manufacture?

6. Determine the maxima and minima of $f$ on the given surface:

(a) The function $f(x, y) = xy$ on the curve $3x^2 + y^2 = 6$.

(b) The function $f(x, y, z) = x + y + 2z$ on the surface $x^2 + y^2 + z^2 = 3$.

(c) The function $f(x, y, z) = x^2 - y^2$ on the surface $x^2 + 2y^2 + 3z^2 = 1$.

3 Answers

1. (a) $(6, 3, 2)$, (b) $(2, \sqrt{2}, 1)$  
2. (a) $(2^{-1/4}, 2^{-1/4}, 2^{1/4})$  
2. (b) $(\pm 1, 0, 0)$  
3. $2 : 4 : 3$  
4. 4 ft. by 6 ft. by 1 ft.  
5. 2 ft. by $\frac{1}{2}$ ft. by $\frac{1}{2}$ ft.  
6. (a) max $\sqrt{3}$, min $-\sqrt{3}$  
(b) max $3\sqrt{2}$, min $-3\sqrt{2}$  
(c) max 1, min $-\frac{1}{2}$