1 Lecture review

1.1 Least Squares Approximation

1. Given points \((x_1, y_1), \ldots, (x_n, y_n)\), the goal is to find the line \(y = ax + b\) that best fits the data.

2. To minimize the total error, we want to minimize the two variable function in \(a, b\) given by

\[
D(a, b) = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2.
\]

To do this, set \(D_a = 0\) and \(D_b = 0\) and solve for \(a, b\).

1.2 Second Derivative Test

1. Given a critical point, how do we tell if it is a local max, a local min, or neither?

2. Compute the following at a critical point \((x_0, y_0)\):

\[
A = f_{xx}(x_0, y_0) \\
\Delta = \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix} = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2.
\]

(a) If \(\Delta > 0\) and \(A > 0\), then \((x_0, y_0)\) is a local min.

(b) If \(\Delta > 0\) and \(A < 0\), then \((x_0, y_0)\) is a local max.

(c) If \(\Delta < 0\), then \((x_0, y_0)\) is a saddle point.

(d) If \(\Delta = 0\), then the test is inconclusive.
2 Problems

1. (2G-1c) Find by the method of least squares the line which best fits the three data points (1, 1), (2, 3), (3, 2).

2. (2H-1) For each of the following functions, find the critical points, and classify them using the second derivative criterion.

   (a) $x^2 - xy - 2y^2 - 3x - 3y + 1$
   (b) $3x^2 + xy + y^2 - x - 2y + 4$
   (c) $2x^4 + y^2 - xy + 1$
   (d) $x^3 - 3xy + y^3$
   (e) $(x^3 + 1)(y^3 + 1)$

3. (2H-6) Two wires of length 4 are cut in the same way into three pieces, of length $x, y$ and $z$; the four $x, y$ pieces are used as the four sides of a rectangle; the two $z$ pieces are bent at the middle and joined at the ends to make a square of side $z/2$.

   (a) Find the rectangle and square made this way which together have the largest and the smallest total area. Using the answer, tell what type the critical point is.
   (b) Confirm the critical point type by using the second derivative test.

3 Answers

1. $y = \frac{1}{2}x + 1$

2. (a) saddle point at (1, -1), (b) local min at (0, 1), (c) saddle point at (0, 0), local min at $\pm \left(\frac{1}{2}, \frac{1}{2}\right)$, (d) saddle point at (0, 0), local min at (1, 1), (e) saddle point at (−1, −1), inconclusive at (0, 0).

3. (a) largest for “point rectangle” and square of side 2, smallest for rectangle of length 4 with zero area and “point square” (b) critical point is a saddle point