1 Lecture review

1.1 Least Squares Approximation

1. Given points \((x_1, y_1), \ldots, (x_n, y_n)\), the goal is to find the line \(y = ax + b\) that best fits the data.

2. To minimize the total error, we want to minimize the two variable function in \(a, b\) given by

\[
D(a, b) = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2.
\]

To do this, set \(D_a = 0\) and \(D_b = 0\) and solve for \(a, b\).

1.2 Second Derivative Test

1. Given a critical point, how do we tell if it is a local max, a local min, or neither?

2. Compute the following at a critical point \((x_0, y_0)\):

\[
A = f_{xx}(x_0, y_0)
\]
\[
\Delta = \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix} = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2.
\]

(a) If \(\Delta > 0\) and \(A > 0\), then \((x_0, y_0)\) is a local min.

(b) If \(\Delta > 0\) and \(A < 0\), then \((x_0, y_0)\) is a local max.

(c) If \(\Delta < 0\), then \((x_0, y_0)\) is a saddle point.

(d) If \(\Delta = 0\), then the test is inconclusive.
2 Problems

1. (2G-1c) Find by the method of least squares the line which best fits the three data points (1, 1), (2, 3), (3, 2).

Solution.
If the least-squares line is \( y = ax + b \), then we seek to minimize the quantity
\[
D = (a + b - 1)^2 + (2a + b - 3)^2 + (3a + b - 2)^2
\]
The partial derivatives of \( D \) are
\[
D_a = 2(14a + 6b - 13)
\]
\[
D_b = 2(6a + 3b - 6)
\]
Setting both equations equal to zero and solving for them results in \( a = 1/2, b = 1 \). Hence the least squares line is
\[
y = \frac{1}{2}x + 1.
\]
2. (2H-1) For each of the following functions, find the critical points, and classify them using the second derivative criterion.

(a) \( x^2 - xy - 2y^2 - 3x - 3y + 1 \)

(b) \( 3x^2 + xy + y^2 - x - 2y + 4 \)

(c) \( 2x^4 + y^2 - xy + 1 \)

(d) \( x^3 - 3xy + y^3 \)

(e) \( (x^3 + 1)(y^3 + 1) \)

Solution.

(a) Solving \( 0 = f_x = 2x - y - 3 \) and \( 0 = f_y = -x - 4y - 3 \) gives the critical point \((1, -1)\). Here, \( f_{xx} = 2 \), \( f_{xy} = -1 \), and \( f_{yy} = -4 \) so \( \Delta = (2)(-4) - (-1)^2 = -9 < 0 \) means that it is a saddle point.

(b) Solving \( 0 = f_x = 6x + y - 1 \) and \( 0 = f_y = x + 2y - 2 \) gives the critical point \((0, 1)\). Here, \( f_{xx} = 6 \), \( f_{xy} = 1 \), and \( f_{yy} = 2 \) so \( \Delta = (6)(2) - (1)^2 = 11 > 0 \) and \( A = 6 > 0 \) means that it is a local min.

(c) Solving \( 0 = f_x = 8x^3 - y \) and \( 0 = f_y = 2y - x \) gives the critical points \((0, 0)\) and \( \pm \left( \frac{1}{2}, \frac{1}{8} \right) \). As \( f_{xx} = 24x^2 \), \( f_{xy} = -1 \), \( f_{yy} = 2 \), we get that the three points are: saddle at \((0, 0)\) (as \( \Delta = -1 < 0 \)), local min at \( \left( \frac{1}{2}, \frac{1}{8} \right) \) (as \( \Delta = 2 > 0 \), \( f_{xx} = 3/2 > 0 \)).

(d) Solving \( 0 = f_x = 3x^2 - 3y \) and \( 0 = f_y = -3x + 3y^2 \) gives the critical points \((0, 0)\) and \((1, 1)\). As \( f_{xx} = 6x \), \( f_{xy} = -3 \), \( f_{yy} = 6y \), we get that the points are: saddle at \((0, 0)\) (as \( \Delta = -9 < 0 \)), local min at \((1, 1)\) (as \( \Delta = 27 > 0 \), \( f_{xx} = 6 > 0 \)).

(e) Solving \( 0 = f_x = 3x^2(y^3 + 1) \) and \( 0 = f_y = 3y^2(x^3 + 1) \) gives the critical points \((0, 0)\) and \((-1, -1)\). As \( f_{xx} = 6x(y^3 + 1) \), \( f_{xy} = 9x^2y^2 \), \( f_{yy} = 6y(x^3 + 1) \), we get that the points are: saddle at \((-1, -1)\) (as \( \Delta = -9 < 0 \)), inconclusive at \((0, 0)\) (as \( \Delta = 0 \)).
3. (2H-6) Two wires of length 4 are cut in the same way into three pieces, of length $x$, $y$ and $z$; the four $x$, $y$ pieces are used as the four sides of a rectangle; the two $z$ pieces are bent at the middle and joined at the ends to make a square of side $z/2$.

(a) Find the rectangle and square made this way which together have the largest and the smallest total area. Using the answer, tell what type the critical point is.

(b) Confirm the critical point type by using the second derivative test.

Solution.

(a) We know that $z = 4 - x - y$ and the total area is

$$f(x, y) = xy + \left(\frac{z}{2}\right)^2 = xy + \frac{1}{4}(4 - x - y)^2.$$ 

The critical points must satisfy

$$0 = f_x = y - \frac{1}{2}(4 - x - y)$$
$$0 = f_y = x - \frac{1}{2}(4 - x - y)$$

Subtracting the two gives $0 = y - x$. Substituting $y = x$ into the first and solving gives $x - \frac{1}{2}(4 - 2x) = 0$, so $x = y = 1$. At this critical point, the total area is $f(1, 1) = (1)(1) + 1^2 = 2$.

Since the region in question is $0 \leq x, 0 \leq y, 0 \leq z = 4 - x - y$, we are finding the extrema in a triangle. This triangle has three boundary pieces.

i. $x = 0, 0 \leq y \leq 4$. Our function is $f(0, y) = (4 - y)^2/4$, which is maximized at $(0, 0)$ (value of 4) and minimized at $(4, 0)$ (value of 0).

ii. $y = 0, 0 \leq x \leq 4$. Our function is $f(x, 0) = (4 - x)^2/4$, which is maximized at $(0, 0)$ (value of 4) and minimized at $(0, 4)$ (value of 0).

iii. $y = 4 - x, 0 \leq x \leq 4$. Our function is $f(x, 4 - x) = x(4 - x) = 4 - (2 - x)^2$ which is maximized at $(2, 2)$ (value of 4) and minimized at $(0, 0)$ and $(4, 4)$ (values of 0).

So we find

$$f(0, 0) = 4 \quad f(4, 0) = 4 \quad f(0, 4) = 4 \quad f(2, 2) = 4 \quad f(1, 1) = 2.$$ 

It follows that the critical point is just a saddle point; to get the maximum total area 4, take either $(x, y, z) = (0, 0, 4)$ or $(2, 2, 0)$, either of which gives a point “rectangle” and a square of side 2; for the minimum total area 0, take $(x, y, z) = (0, 4, 0)$ or $(4, 0, 0)$, which gives a “rectangle” of length 4 with zero area, and a point square.

(b) We have $f_{xx} = 1/2, f_{xy} = 3/2, f_{yy} = 1/2$ for all $x, y$ so $\Delta = (1/2)^2 - (3/2)^2 = -2 < 0$ means that $(1, 1)$ is a saddle point.