1 Lecture review

1.1 Tangent planes and linear approximations

1. The tangent plane to \( z = f(x, y) \) at the point \((x_0, y_0, z_0)\) is

\[
z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

The right hand side of this expression is sometimes called the linear approximation to \( f(x, y) \) at the point \((x_0, y_0)\):

\[
L(x, y) = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

2. Near the point \((x_0, y_0, z_0)\), the equation of the tangent plane approximates the function \( f(x, y) \):

\[
f(x, y) \approx L(x, y) = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

3. The intuition is as follows: near \((x_0, y_0)\), a change in \( x \) by \( \Delta x \) (assuming \( \Delta x \) is small) induces a change in \( f(x, y) \) by approximately \( f_x(x_0, y_0)\Delta x \). Similarly, a change in \( y \) by \( \Delta y \) (assuming \( \Delta y \) is small) induces a change in \( f(x, y) \) by approximately \( f_y(x_0, y_0)\Delta y \).

In other words, if we use \( \Delta f \) to represent the change in \( f \), we have

\[
\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.
\]
2 Problems

1. (2B-1) Give the equation of the tangent plane to each of these surfaces at the point indicated.

   (a) $z = xy^2$ at $(1, 1, 1)$. Use this tangent plane to approximate the value of $z$ when $x = 1.01$, $y = 1.01$. How close is your approximation to the true value?

   (b) $z = y^2/x$ at $(1, 2, 4)$. Use this tangent plane to approximate the value of $z$ when $x = 1.01$, $y = 2.01$. How close is your approximation to the true value?

Solution.

(a) Since $z_x = y^2$ and $z_y = 2xy$, we have that at $(x, y, z) = (1, 1, 1)$, the partial derivatives are $(z_x, z_y) = (1, 2)$. Hence the tangent plane is

$$z - 1 = (x - 1) + 2(y - 1)$$

meaning that the approximate value of $z$ is

$$z - 1 \approx (0.01) + 2(0.01) \implies z \approx 1.03.$$  

The actual value is 1.030301.

(b) Since $z_x = -y^2/x^2$ and $z_y = 2y/x$ we have that at $(x, y, z) = (1, 2, 4)$, the partial derivatives are $(z_x, z_y) = (-4, 4)$. Hence the tangent plane is

$$z - 4 = -4(x - 1) + 4(y - 2)$$

meaning that the approximate value of $z$ is

$$z - 4 \approx (-4)(0.01) + 4(0.01) \implies z \approx 4.$$  

The actual value is 4.000099.
2. (2B-3) Using the approximation formula, find the approximate change in the hypotenuse of a right triangle, if the legs, initially of length 3 and 4, are each increased by .010. Compute the actual increase and compare it to the approximation.

Solution.
If the legs are $x$ and $y$, and the hypotenuse is $z$, then the Pythagorean theorem gives

$$z = \sqrt{x^2 + y^2}.$$ 

Hence

$$z_x = \frac{x}{\sqrt{x^2 + y^2}}$$

$$z_y = \frac{y}{\sqrt{x^2 + y^2}}$$

So at the point $(x, y) = (3, 4)$ the tangent plane is

$$z - 5 = \frac{3}{5}(x - 3) + \frac{4}{5}(y - 4)$$

Hence the approximate change is

$$\frac{3}{5}(0.010) + \frac{4}{5}(0.010) = 0.014.$$

The actual increase is

$$\sqrt{4.010^2 + 3.010^2} - \sqrt{4^2 + 3^2} \approx 0.0140004.$$
3. (2B-6) To determine the volume of a cylinder of radius around 2 and height around 3, about how accurately should the radius and height be measured for the error in the calculated volume not to exceed 0.1?

Solution.
We have

\[ V = \pi r^2 h \]

so

\[ V_r = 2\pi rh \]
\[ V_h = \pi r^2, \]

so when \((r, h) = (2, 3)\) we have \((V_r, V_h) = (12\pi, 4\pi)\). Therefore around \((r, h) = (2, 3)\) we know

\[ \Delta V \approx 12\pi \Delta r + 4\pi \Delta h \]

If the maximal error in \(r\) (which is \(\Delta r\)) and the maximal error in \(h\) (which is \(\Delta h\)) is \(\varepsilon\), then the actual volume will have error at most

\[ 12\pi \varepsilon + 4\pi \varepsilon = 16\pi \varepsilon. \]

Since this is at most 0.1, we see that

\[ \varepsilon < \frac{0.1}{16\pi} \approx 0.002. \]
4. Consider the function \( z = 3x + 4y \). Compute the tangent plane at the point \((1, 1)\). What do you notice?

\textit{Solution.}

Computing \( z_x \) and \( z_y \) gives \( z_x = 3, \ z_y = 4 \). The surface goes through \((1, 1, 7)\), so the equation of the tangent plane is

\[ z - 7 = 3(x - 1) + 4(y - 1) \]

Adding 7 to both sides gives that the equation of the tangent plane is

\[ z = 3x + 4y. \]

So the tangent plane is literally the plane we started with.
5. Suppose the tangent plane to a surface given by \( z = f(x, y) \) at the point \( P \) given by \( x = 1, y = 2 \) is \( x + 2y + 3z = 4 \).

(a) What is \( f(1, 2) \)?   (b) What is \( f_x(1, 2) \)?   (c) What is \( f_y(1, 2) \)?

**Solution.**

Rewriting the equation of the plane as

\[
z + \frac{1}{3} = -\frac{1}{3}(x - 1) - \frac{2}{3}(y - 2),
\]

we can read off immediately the following values:

\[
\begin{align*}
  f(1, 2) &= -\frac{1}{3} \\
  f_x(1, 2) &= -\frac{1}{3} \\
  f_y(1, 2) &= -\frac{2}{3}.
\end{align*}
\]
6. Consider the function \( z = \sqrt{9 - x^2 - y^2} \).

(a) Describe this shape.

(b) Using geometric intuition, what would be a unit normal vector to this surface at the point \((2, 1, 2)\)?

(c) Using partial differentiation, verify that your answer in the previous part is correct. Compute the tangent plane at the point \((2, 1, 2)\).

**Solution.**

(a) Rewriting this as \( x^2 + y^2 + z^2 = 9 \), we see that this is the upper half of a sphere of radius 3 centered at the origin.

(b) A vector pointing away from the origin would be normal to the sphere, which is

\[
\frac{(2, 1, 2) - (0, 0, 0)}{||(2, 1, 2) - (0, 0, 0)||} = \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle.
\]

The other unit normal vector is the one pointing to the origin, which would be \(-\left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle\).

(c) We have

\[
z_x = \frac{-x}{\sqrt{9 - x^2 - y^2}}
\]

\[
z_y = \frac{-y}{\sqrt{9 - x^2 - y^2}}
\]

and hence at \((x, y, z) = (2, 1, 2)\) we see that \((z_x, z_y) = (-1, -\frac{1}{2})\). Therefore the tangent plane at \((2, 1, 2)\) would be

\[
z - 2 = -(x - 2) - \frac{1}{2}(y - 1).
\]
7. Consider the function \( z = \sqrt{x^2 + y^2} \).

(a) Describe the shape.

(b) Compute the equation of the tangent plane at an arbitrary point. This plane intersects the shape at more than just that point; what is the intersection?

(c) What happens at the origin when you attempt to compute the tangent plane?

Solution.

(a) This shape is a cone passing through the origin whose axis of symmetry is the \( z \)-axis.

(b) We have

\[
\begin{align*}
    z_x &= \frac{x}{\sqrt{x^2 + y^2}} \\
    z_y &= \frac{y}{\sqrt{x^2 + y^2}}
\end{align*}
\]

At the point \((x_0, y_0, z_0)\) the equation of the tangent plane is

\[
z - z_0 = \frac{x_0}{\sqrt{x_0^2 + y_0^2}}(x - x_0) + \frac{y_0}{\sqrt{x_0^2 + y_0^2}}(y - y_0) = \frac{x_0}{z_0}(x - x_0) + \frac{y_0}{z_0}(y - y_0).
\]

Since \(z_0^2 = x_0^2 + y_0^2\), this can also be written as

\[
z = \frac{x_0}{z_0}x + \frac{y_0}{z_0}y.
\]

To intersect this with the cone, we substitute \( z = \sqrt{x^2 + y^2} \) and square both sides:

\[
x^2 + y^2 = \left(\frac{x_0}{z_0}x + \frac{y_0}{z_0}y\right)^2.
\]

Rearranging this gives

\[
0 = (x^2 + y^2) - \left(\frac{x_0}{z_0}x + \frac{y_0}{z_0}y\right)^2
\]

\[
= \left(\frac{y_0}{z_0}x - \frac{x_0}{z_0}y\right)^2,
\]

so solving this gives \( y_0 x = x_0 y \). In other words, the intersection is the line \( \frac{x}{x_0} = \frac{y}{y_0} = \frac{1}{z_0} \); i.e, the line through the point \((x_0, y_0, z_0)\) and the origin.

(c) The origin is the “cone point” of the cone, so there is no well-defined tangent plane. If we try evaluating \( z_x \) or \( z_y \) at the origin, the denominator is 0.