1 Lecture review

1.1 Lines in 3D

1. Parametric form of a line in 3D: \( \mathbf{r} = (x_0, y_0, z_0) + t \mathbf{a} \) gives the equation of a line going through the point \((x_0, y_0, z_0)\) in the direction of \( \mathbf{a} \).

2. Symmetric form of a line in 3D: \( \frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3} \) gives the equation of a line going through the point \((x_0, y_0, z_0)\) in the direction of \((a_1, a_2, a_3)\).

1.2 Planes in 3D

1. If a plane passes through \((x_0, y_0, z_0)\) and \( \mathbf{n} \) is a normal vector to the plane, then its equation is \( \mathbf{n} \cdot (x - x_0, y - y_0, z - z_0) = 0 \).

2. To determine \( \mathbf{n} \), take the cross product of two vectors lying in the plane.

3. To determine the distance from a point \(Q\) to a plane \(P\), take any point \(S\) of \(P\) and take the component of \(\mathbf{QS}\) in the direction of the normal to \(P\).

1.3 Intersections

1. In 2D, the only way for 2 lines to not intersect is for them to be parallel.

2. In 3D, the only way for 2 lines to not intersect is for them to be parallel or skew (lie in parallel planes).

3. In 3D, the only way for a line \(L\) and a plane \(P\) to not intersect is if \(L\) lies in a plane parallel to \(P\).
   
   (a) To find the intersection, plug the parametric equations of \(L\) into the equation for \(P\).

4. In 3D, the only way for two planes to not intersect is if they are parallel.
   
   (a) If they intersect in a line, then the direction of the line is determined by the cross product of the two normal vectors. The angle of the intersection is the angle between the normal vectors.

1.4 Parametric Equations

1. A parametric equation is one of the form \( \mathbf{r}(t) = (x(t), y(t), z(t)) \). This is a vector-valued function of \(t\): the input is \(t\) and the output is the vector \(\mathbf{r}(t)\).

2. The velocity is \( \mathbf{v}(t) = \mathbf{r}'(t) = (x'(t), y'(t), z'(t)) \).

3. The speed is \( \|\mathbf{v}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \).

4. The acceleration is \( \mathbf{a}(t) = \mathbf{v}'(t) = (x''(t), y''(t), z''(t)) \).
2 Problems

1. Compute the intersection and angle of intersection between
   (a) (i) the line going through (4, 0, 4), (1, 2, 3) and (ii) the line \( x = 2(y + 1) = z \).
   (b) (i) the line going through (0, 2, 3), (2, 0, 1) and (ii) the plane \( x - y = 6 \).
   (c) (i) the plane \( x + y + z = 3 \) and (ii) the plane \( 3x + y - 2z = 2 \).

2. Check that the following objects do not intersect. Then find the distance between them.
   (a) (i) the point (0, 2, 3) and (ii) the plane \( 2x + 3y - 6z = -1 \).
   (b) (i) the plane \( 2x + 3y - 6z = 0 \) and (ii) the plane \( 2x + 3y - 6z = 2 \).
   (c) (i) the line \( 4x = 4y = z \) and (ii) the plane \( 2x + 2y - z = 1 \).
   (d) (i) the point (0, 0, 0) and (ii) the line \( x = 4y - 5 = 11 - 4z \).
   (e) (i) the line \( x = 4y = 4z \) and (ii) the line \( x - 3 = 2(y - 1) = -z \).
   (f) (i) the line through (5, 2, 11) in the direction \( \langle 3, -4, 2 \rangle \) and (ii) the line through (1, 16, 16) in the direction \( \langle 3, -4, 2 \rangle \).

3. (II-3c) Describe the motion given by \( \mathbf{r} = (t^2 + 1) \mathbf{i} + t^3 \mathbf{j} \) by: (i) giving the \( xy \)-equation of the curve and (ii) telling what part of the curve is actually traced out.

4. (II-4) A roll of plastic tape of outer radius \( a \) is held in a fixed position while the tape is being unwound counterclockwise. The end \( P \) of the unwound tape is always held so the unwound portion is perpendicular to the roll. Taking the center of the roll to be the origin \( O \), and the end \( P \) to be initially at \( (a, 0) \), write parametric equations for the motion of \( P \). (Use vectors; express the position vector \( \overrightarrow{OP} \) as a vector function of one variable.)

5. (II-5) A string is wound clockwise around the circle of radius \( a \) centered at the origin \( O \); the initial position of the end \( P \) of the string is \( (a, 0) \). Unwind the string, always pulling it taut (so it stays tangent to the circle). Write parametric equations for the motion of \( P \). (Use vectors; express the position vector \( \overrightarrow{OP} \) as a vector function of one variable.)

3 Answers

1. (a) (2, 0, 2) at an angle of \( \arccos(\sqrt{6}/3) \), (b) (4, -2, -1) at an angle of \( \arccos(\sqrt{3}/3) \), (c) intersection line is \( x = 1 - 3t \), \( y = 1 + 5t \), \( z = 1 - 2t \) (equivalently, \( x = \frac{6}{3} = \frac{y - 1}{5} = \frac{z - 1}{2} \)) and angle is \( \arccos(\sqrt{12}/21) \)
2. (a) 1/7, (b) 2/7, (c) 1/3, (d) 3, (e) 5/3, (f) 11
3. (i) \( y^2 = (x - 1)^3 \); (ii) all of it
4. \( \mathbf{r}(\theta) = a(1 + \theta)(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) \)
5. \( \mathbf{r}(\theta) = a(\cos \theta + \theta \sin \theta) \mathbf{i} + a(\sin \theta - \theta \cos \theta) \mathbf{j} \)