1 Lecture review

1.1 Cross Product

1. The cross product is \( \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \).

2. \( \mathbf{a} \times \mathbf{b} \) is a 3D vector of length \(|\mathbf{a}||\mathbf{b}|\sin \theta\) (which is the area of the parallelogram formed by \( \mathbf{a} \) and \( \mathbf{b} \)) which points in the normal direction to \( \mathbf{a} \) and \( \mathbf{b} \). Choice of normal direction is determined by the right hand rule.

3. \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \) volume of parallelepiped formed by \( \mathbf{a}, \mathbf{b}, \mathbf{c} \).

1.2 Lines in 2D

1. Parametric form of a line in 2D: \( \mathbf{r} = (x_0, y_0) + t \mathbf{a} \) gives the equation of a line going through the point \((x_0, y_0)\) in the direction of \( \mathbf{a} \).

2. Another way to describe a line in 2D: take a normal vector \( \mathbf{n} \) and a point \((x_0, y_0)\) on the line. Then the equation of the line is \( \mathbf{n} \cdot (x - x_0, y - y_0) = 0 \).

1.3 Lines in 3D

1. Parametric form of a line in 3D: \( \mathbf{r} = (x_0, y_0, z_0) + t \mathbf{a} \) gives the equation of a line going through the point \((x_0, y_0, z_0)\) in the direction of \( \mathbf{a} \).

2. Symmetric form of a line in 3D: \( \frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3} \) gives the equation of a line going through the point \((x_0, y_0, z_0)\) in the direction of \((a_1, a_2, a_3)\).

1.4 Planes in 3D

1. If a plane passes through \((x_0, y_0, z_0)\) and \( \mathbf{n} \) is a normal vector to the plane, then its equation is \( \mathbf{n} \cdot (x - x_0, y - y_0, z - z_0) = 0 \).

2. To determine \( \mathbf{n} \), take the cross product of two vectors lying in the plane.

3. To determine the distance from a point \( Q \) to a plane \( P \), take any point \( S \) of \( P \) and take the component of \( \overrightarrow{QS} \) in the direction of the normal to \( P \).

1.5 Intersections

1. In 2D, the only way for 2 lines to not intersect is for them to be parallel.

2. In 3D, the only way for 2 lines to not intersect is for them to be parallel or skew (lie in parallel planes).

3. In 3D, the only way for a line \( L \) and a plane \( P \) to not intersect is if \( L \) lies in a plane parallel to \( P \).

   (a) To find the intersection, plug the parametric equations of \( L \) into the equation for \( P \).

4. In 3D, the only way for two planes to not intersect is if they are parallel.

   (a) If they intersect in a line, then the direction of the line is determined by the cross product of the two normal vectors. The angle of the intersection is the angle between the normal vectors.
2 Problems

1. Find the equation of
   (a) the line passing through (1, 0, 0) and (0, 1, 1).
   (b) the line that intersects the plane \( x + 2y + 3z = 6 \) at a 90 degree angle at the point (3, 0, 1).
   (c) the plane passing through (1, 1, 1), (5, 0, -1), and (0, 1, 2).
   (d) the plane that intersects the line \( x = 2y = 3z \) at a 90 degree angle at the point (6, 3, 2).

2. Compute the intersection and angle of intersection between
   (a) (i) the line going through (4, -4, 0), (1, 2, 3) and (ii) the line \( x = 2(y + 1) = z \).
   (b) (i) the line going through (0, 2, 3), (2, 0, 1) and (ii) the plane \( x - y = 6 \).
   (c) (i) the plane \( x + y + z = 3 \) and (ii) the plane \( 3x + y - 2z = 2 \).

3. Check that the following objects do not intersect. Then find the distance between them.
   (a) (i) the point (0, 2, 1) and (ii) the plane \( 2x + 3y - 6z = -1 \).
   (b) (i) the plane \( 2x + 3y - 6z = 0 \) and (ii) the plane \( 2x + 3y - 6z = 2 \).
   (c) (i) the line \( 4x = 4y = z \) and (ii) the plane \( 2x + 2y - z = 1 \).
   (d) (i) the point (0, 0, 0) and (ii) the line \( x = 4y - 5 = 11 - 4z \).
   (e) (i) the line \(-x = 4y = 4z \) and (ii) the line \( x - 3 = 2(y - 1) = -z \).
   (f) (i) the line through (5, 2, 11) in the direction \( \langle 3, -4, 2 \rangle \) and (ii) the line through (1, 16, 16) in the direction \( \langle 3, -4, 2 \rangle \).

3 Answers

1. (a) \( x = 1 - t, \ y = t, \ z = t \) or \( 1 - x = y = z \), (b) \( x = 3 + t, \ y = 2t, \ z = 1 + 3t \) or \( x - 3 = \frac{y}{2} = \frac{z - 1}{3} \),
   (c) \( x + 2y + z = 4 \), (d) \( 6x + 3y + 2z = 49 \)

2. (a) \( 2, 0, 2 \) at an angle of arccos(\( \sqrt{6}/9 \)), (b) \( 4, -2, -1 \) at an angle of arccos(\( \sqrt{3}/3 \)), (c) intersection line is \( x = 1 - 3t, \ y = 1 + 5t, \ z = 1 - 2t \) (equivalently, \( \frac{1 - z}{3} = \frac{y - 1}{5} = \frac{x - 1}{2} \)) and angle is arccos(\( \sqrt{12}/21 \))

3. (a) 1/7, (b) 2/7, (c) 1/3, (d) 3, (e) 5/3, (f) 11