1 Lecture review

1.1 Dot products

1. The dot product of two vectors is a scalar. If \( \mathbf{v} = (a_1, a_2, a_3) \) and \( \mathbf{w} = (b_1, b_2, a_3) \), then
\[
\mathbf{v} \cdot \mathbf{w} = a_1 b_1 + a_2 b_2 + a_3 b_3.
\]

2. It satisfies the following properties
   (a) \( \mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2 \).
   (b) \( \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \).
   (c) \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \).

3. If \( \theta \) is the angle between vectors \( \mathbf{a} \) and \( \mathbf{b} \), then
\[
\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \cos \theta.
\]

4. If \( \mathbf{a} \) and \( \mathbf{b} \) point in the same direction (\( \theta = 0 \)), then \( \mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \).

5. If \( \mathbf{a} \) and \( \mathbf{b} \) are perpendicular, then \( \mathbf{a} \cdot \mathbf{b} = 0 \).

6. The component of \( \mathbf{b} \) in the direction of \( \mathbf{a} \) is \( ||\mathbf{a}|| \cos \theta \).

7. The projection of \( \mathbf{b} \) in the direction of \( \mathbf{a} \) is \( ||\mathbf{b}|| \cos \theta (\mathbf{a}/||\mathbf{a}||) = (\mathbf{a} \cdot \mathbf{b})\mathbf{a}/||\mathbf{a}||^2 \).

1.2 Cross products and Determinants

1. Determinants (2 dimensions).
   (a) \( \det(\mathbf{a}, \mathbf{b}) = ||\mathbf{a}|| ||\mathbf{b}|| \sin \theta \), where \( \theta \) is the counterclockwise angle from \( \mathbf{a} \) to \( \mathbf{b} \). This gives the signed area of the parallelogram formed by \( \mathbf{a} \) and \( \mathbf{b} \).
   (b) If \( \mathbf{a} = (a_1, a_2) \) and \( \mathbf{b} = (b_1, b_2) \), then
\[
\det(\mathbf{a}, \mathbf{b}) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.
\]

2. Determinants (3 dimensions).
   (a) \( \det(\mathbf{a}, \mathbf{b}, \mathbf{c}) \) is the signed volume of the parallelepiped formed from the vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \).
   (b) We have
\[
\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.
\]

3. Cross products (3 dimensions).
   (a) To compute the cross product of \( \mathbf{a} = (a_1, a_2, a_3) \) and \( \mathbf{b} = (b_1, b_2, b_3) \), we have
\[
\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.
\]
   (b) \( \mathbf{a} \times \mathbf{b} \) is a 3D vector of length \( ||\mathbf{a}|| \cdot ||\mathbf{b}|| \sin \theta \) pointing in the normal direction to \( \mathbf{a} \) and \( \mathbf{b} \). Choice of normal direction is determined by the right hand rule.
   (c) \( \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \).
   (d) \( \mathbf{a} \times \mathbf{b} = 0 \) when \( \mathbf{a} \) and \( \mathbf{b} \) point in the same direction.
2 Problems

1. What are the $i$ $j$-components of a plane vector $A$ of length 3, if it makes an angle of $30^\circ$ with $i$ and $60^\circ$ with $j$. Is the second condition redundant?

Solution.

Set $A = a_1i + a_2j$. Then

$$a_1 = A \cdot i = 3 \cos 30^\circ = \frac{3\sqrt{3}}{2}, \quad a_2 = A \cdot j = 3 \cos 60^\circ = \frac{3}{2}.$$ 

Thus

$$A = \frac{3\sqrt{3}}{2} i + \frac{3}{2} j.$$ 

The second condition is not redundant since there are two vectors of length 3 making an angle of $30^\circ$ with $i$. 


2. A small plane wishes to fly due north at 200 mph (as seen from the ground), in a wind blowing from the northeast at 50 mph. Tell with what vector velocity in the air it should travel (give the $i\ j$-components).

Solution.

The unit vector for the wind blowing from the northeast is $(-\hat{i} - \hat{j})/\sqrt{2}$. Thus the vector for the wind is $\mathbf{w} = 50(-\hat{i} - \hat{j})/\sqrt{2}$. We want to find the vector velocity $\mathbf{v}$ for the small plane which satisfies

$$\mathbf{v} + \mathbf{w} = 200\hat{j} \quad \implies \quad \mathbf{v} = \frac{50}{\sqrt{2}}\hat{i} + \left( 200 + \frac{50}{\sqrt{2}} \right)\hat{j}.$$
3. Find the angle between the vectors $\mathbf{i} - \mathbf{k}$ and $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

*Solution.*

The magnitude of $\mathbf{i} - \mathbf{k}$ is $\sqrt{2}$ and the magnitude of $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ is 6. Then

$$
\cos \theta = \frac{(\mathbf{i} - \mathbf{k}) \cdot (4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})}{\sqrt{2} \cdot 6} = \frac{4 + 2}{\sqrt{2} \cdot 6} = \frac{1}{\sqrt{2}} \implies \theta = \frac{\pi}{4}.
$$
4. Using vectors, find the angle between a longest diagonal $PQ$ of a cube, and

(a) a diagonal $PR$ of one of its faces
(b) an edge $PS$ of the cube

Solution.

Place the cube in the first octant so the origin is at one corner $P$, and $i, j, k$ are three edges. The longest diagonal $PQ = i + j + k$, a face diagonal $PR = i + j$, and an edge $PS = i$.

\[
\begin{align*}
(a) \quad \cos \theta &= \frac{PQ \cdot PR}{|PQ||PR|} = \frac{2}{\sqrt{3}\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{3}} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right). \\
(b) \quad \cos \theta &= \frac{PQ \cdot PS}{|PQ||PS|} = \frac{1}{\sqrt{3}}, \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right). 
\end{align*}
\]
5. Find $\mathbf{A} \times \mathbf{B}$ if

(a) $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}, \; \mathbf{B} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

(b) $\mathbf{A} = 2\mathbf{i} - 3\mathbf{k}, \; \mathbf{B} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

Solution.

(a) $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & -1 & -1 \end{vmatrix} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} = 3(\mathbf{i} + \mathbf{j} + \mathbf{k})$

(b) $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -3 \\ 1 & 1 & -1 \end{vmatrix} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$
6. Find the area of the triangle in space having its vertices at the points

\[ P : (2, 0, 1), \quad Q : (3, 1, 0), \quad R : (-1, 1, -1). \]

**Solution.**

The area of the \( PQR \) is half the area of the parallelogram with edges \( PQ = (1, 1, -1), \) \( PR = (-3, 1, -2). \) The latter area is \( |PQ \times PR| \). We compute

\[
PQ \times PR = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -3 & 1 & -2 \end{vmatrix} = -\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}.
\]

Thus

\[
\text{area of } PQR = \frac{1}{2} \|PQ \times PR\| = \frac{1}{2} \sqrt{1 + 25 + 16} = \frac{\sqrt{42}}{2}.
\]
7. Show using vector methods that the diagonals of a rectangle have equal length.

Solution.

Place a rectangle $PQRS$ with side lengths $a$ and $b$ in the plane so the origin is at one corner $P$, and $\mathbf{a}, \mathbf{b}$ are two edges. We have one diagonal $PR = a\mathbf{i} + b\mathbf{j}$. We have another diagonal $QS = -a\mathbf{i} + b\mathbf{j}$. We can see that

$$|PR| = \sqrt{a^2 + b^2} = |QS|.$$
8. Show using vector methods that if the diagonals of a parallelogram have equal length, then it must be a rectangle.

_solution_.

Consider a parallelogram $OPQR$ with $O$ at the origin. Let $\mathbf{a} = OP$ and $\mathbf{b} = OR$. Then the diagonals of this parallelogram are $\mathbf{a} + \mathbf{b} = OQ$ and $\mathbf{a} - \mathbf{b} = RP$.

If the diagonals have equal length, then

$$\|\mathbf{a} + \mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\|.$$

Squaring both sides gives

$$\|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a} - \mathbf{b}\|^2.$$

Since $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$, it follows that

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}).$$

Expanding and cancelling like terms from both sides gives

$$2(\mathbf{a} \cdot \mathbf{b}) = -2(\mathbf{a} \cdot \mathbf{b}).$$

So we see that $\mathbf{a} \cdot \mathbf{b} = 0$, meaning that the angle between $\mathbf{a}$ and $\mathbf{b}$ is 90 degrees, so that the parallelogram is a rectangle.