1 Lecture review

1.1 Applications of double integration

1. \( \iint_R 1 \, dA \) gives the area of \( R \).

2. \( \iint_R f(x, y) - g(x, y) \, dA \) gives the volume between the surfaces \( z = f(x, y) \) and \( z = g(x, y) \) lying over the region \( R \).

3. \( \frac{1}{\text{Area}(R)} \iint_R f(x, y) \, dA \) gives the average value of a function \( f(x, y) \) over a region \( R \).

4. The mass and center of mass of a thin metal plate \( R \) with density function \( \rho(x, y) \) are

\[
    m = \iint_R \rho(x, y) \, dA \quad x_0 = \frac{1}{m} \iint_R x \rho(x, y) \, dA \quad y_0 = \frac{1}{m} \iint_R y \rho(x, y) \, dA
\]

1.2 Double integrals in polar coordinates

1. Conversion formulas:

\[
    x = r \cos \theta \quad r = \sqrt{x^2 + y^2} \\
    y = r \sin \theta \quad \theta = \tan^{-1}(y/x).
\]

2. Conversion of integral from rectangular to polar:

\[
    \int_{x, y \text{ bounds}} f(x, y) \, dy \, dx = \int_{r, \theta \text{ bounds}} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.
\]

Warning: Do not forget the extra “\( r \)” that appears! Sometimes it makes the difference between being able to evaluate the integral and not.

3. Analogous to vertical and horizontal strips, now one has “radial strips” and “azimuthal” strips, depending on the order of integration.

   (a) Radial: \( \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} (\ldots) \, dr \, d\theta \). That is, as \( \theta \) goes from \( \theta_1 \) to \( \theta_2 \), \( r \) is going from \( r_1(\theta) \) to \( r_2(\theta) \).

   This is usually the order that is easiest to compute.

   (b) Azimuthal: \( \int_{r_1}^{r_2} \int_{\theta_1(r)}^{\theta_2(r)} (\ldots) \, d\theta \, dr \). That is, as \( r \) goes from \( r_1 \) to \( r_2 \), \( \theta \) is going from \( \theta_1(r) \) to \( \theta_2(r) \).

   It is uncommon for this order to be easier to compute.
2 Problems

1. (3B-1) Express each double integral over the given region $R$ as an iterated integral in polar coordinates.

(a) The region lying inside the circle with center at the origin and radius 2 and to the left of the vertical line through $(-1,0)$.
(b) The circle of radius 1, and center at $(0,1)$.
(c) The finite region bounded by the $y$-axis, the line $y = a$, and a quarter of the circle of radius $a$ and center at $(a,0)$.

2. (3B-2) Evaluate the following using polar coordinates.

(a) $\int\int_{R} \frac{dA}{r}; R$ is the first quadrant loop of $r \leq \sin 2\theta$.
(b) $\int\int_{R} \frac{dx \, dy}{1 + x^2 + y^2}; R$ is the first quadrant portion of $x^2 + y^2 \leq a^2$.
(c) $\int\int_{R} \frac{dA}{\sqrt{1 - x^2 - y^2}}; R$ is the right half disk of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$.

3. (3B-3) Find the volumes of the following using polar coordinates.

(a) a solid hemisphere of radius $a$
(b) the domain under $z = xy$ lying over the first quadrant quarter-disc of radius $a$
(c) the domain under $z = \sqrt{x^2 + y^2}$ lying over the circle of radius 1 centered at $(0,1)$
(d) the domain under $z = x^2 + y^2$ lying over the right hand loop of $r^2 = \cos \theta$

4. (3C-2,4) Find the center of mass for

(a) the region inside $\sin(x)$ for $0 \leq x \leq \pi$ if (i) $\delta = 1$, (ii) $\delta = y$.
(b) the sector of a circle of radius $a$ whose corners are $(a \cos \alpha, -a \sin \alpha), (0,0), (a \cos \alpha, a \sin \alpha)$ if $\delta = 1$.

5. Find the center of mass for the following objects.

(a) the interior of the cardoid $r = 2a(1 - \cos \theta)$ with density $1/r$
(b) the square with vertices $(0,0), (0,a), (a,a), (a,0)$ with density $x+y$
(c) a triangular plate bounded by the lines $x = 0, y = 0, x + y = a$ with density $x$

3 Answers

1. (a) $\int_{\pi/2}^{\pi/3} \int_{\sec \theta}^{\tan \theta} d\theta
d\theta$, (b) $\int_{0}^{\pi} \int_{0}^{2\sin \theta} d\theta
d\theta$, (c) $\int_{0}^{\pi} \int_{0}^{a} d\theta
d\theta$, (d) $\int_{\pi/4}^{\pi/2} \int_{2a \cos \theta}^{a \cos \theta} d\theta
d\theta$

2. (a) 1, (b) $\frac{3}{4} \ln(1 + a^2)$, (c) $\frac{1}{2} \pi - 1$

3. (a) $\frac{2}{3} \pi a^3$, (b) $\frac{1}{2} a^4$, (c) $\frac{32}{9}$, (d) $\frac{1}{5} \pi$

4. (a) (i) $\left(\frac{1}{2} \pi, \frac{1}{8} \pi\right)$, (ii) $\left(\frac{1}{2} \pi, \frac{10}{9} \pi\right)$, (b) $\left(\frac{2a}{3a} \sin \alpha, 0\right)$

5. (a) $(-a,0)$, (b) $\left(\frac{5a}{12}, \frac{7a}{12}\right)$, (c) $\left(\frac{1}{2} a, \frac{1}{4} a\right)$