1 Lecture review

1.1 Double integration

<table>
<thead>
<tr>
<th>Single variable</th>
<th>Two variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = f(x) )</td>
<td>( z = f(x, y) )</td>
</tr>
<tr>
<td>Interval ( I = [a, b] )</td>
<td>Region ( R )</td>
</tr>
<tr>
<td>Area under the curve ( y = f(x) ) is</td>
<td>Volume under the surface ( z = f(x, y) ) is</td>
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<tr>
<td>( \int_I f(x) , dx = \int_a^b f(x) , dx. )</td>
<td>( \iint_R f(x, y) , dA. )</td>
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</table>

1. If the region \( R \) is the rectangular box given by \( a \leq x \leq b, c \leq y \leq d \), then

\[
\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy
\]

(a) Swapping the order of integration is allowed, but make sure to also swap the bounds.

2. If the region is not rectangular, choose the bounds of integration more carefully. You can either

(i) find \( y_B(x) \), \( y_T(x) \) such that \( R \) is given by \( a \leq x \leq b, y_B(x) \leq y \leq y_T(x) \) so that

\[
\iint_R f(x, y) \, dA = \int_a^b \int_{y_B(x)}^{y_T(x)} f(x, y) \, dy \, dx
\]

(Vertical Strips)

(ii) find \( x_L(y) \), \( x_R(y) \) such that \( R \) is given by \( c \leq y \leq d, x_L(y) \leq x \leq x_R(y) \) so that

\[
\iint_R f(x, y) \, dA = \int_c^d \int_{x_L(y)}^{x_R(y)} f(x, y) \, dx \, dy
\]

(Horizontal Strips)

(a) Sometimes one of these is very hard/impossible while the other is more tractable; if you get stuck doing vertical strips, make sure to try horizontal (and vice versa).

1.2 Applications of Double Integration

1. \( \iint_R 1 \, dA \) gives the area of \( R \).

2. \( \iint_R f(x, y) - g(x, y) \, dA \) gives the volume between the surfaces \( z = f(x, y) \) and \( z = g(x, y) \) lying over the region \( R \).

3. \( \frac{1}{\text{Area}(R)} \iint_R f(x, y) \, dA \) gives the average value of a function \( f(x, y) \) over a region \( R \).

4. The mass and center of mass of a thin metal plate \( R \) with density function \( \rho(x, y) \) are

\[
m = \iint_R \rho(x, y) \, dA \quad x_0 = \frac{1}{m} \iint_R x \rho(x, y) \, dA \quad y_0 = \frac{1}{m} \iint_R y \rho(x, y) \, dA
\]
2 Problems

1. (3A-1) Evaluate the following iterated integrals

\[ \begin{align*}
& (b) \quad \int_0^{\pi/2} \int_0^\pi (u \sin t + t \cos u) \, dt \, du \\
& (d) \quad \int_0^1 \int_0^u \sqrt{u^2 + 4} \, dv \, du
\end{align*} \]

Solution.

(b) We have

\[ \begin{align*}
\int_0^{\pi/2} \int_0^\pi (u \sin t + t \cos u) \, dt \, du &= \int_0^{\pi/2} \left[-u \cos t + \frac{1}{2} t^2 \cos u \right]_t=0^\pi \, du \\
&= \int_0^{\pi/2} 2u + \frac{1}{2} \pi^2 \cos u \, du \\
&= \left[u^2 + \frac{1}{2} \pi^2 \sin u \right]_0^{\pi/2} \\
&= \frac{3}{4} \pi^2.
\end{align*} \]

(d) We have

\[ \begin{align*}
\int_0^1 \int_0^u \sqrt{u^2 + 4} \, dv \, du &= \int_0^1 \left[v \sqrt{u^2 + 4} \right]_v=0^u \, du \\
&= \int_0^1 u(u^2 + 4) \, du \\
&= \left[(u^2 + 4)^{3/2} \right]_0^1 \\
&= \frac{1}{3} (5\sqrt{5} - 8).
\end{align*} \]
2. (3A-2) For the given regions $R$, express the following as iterated integrals (using the given order of integration): (i) $\int \int_R d\,y \,d\,x$, (ii) $\int \int_R d\,x \,d\,y$.

(a) $R$ is the triangle with vertices at the origin, $(0, 2)$, $(-2, 2)$.
(c) $R$ is the sector of the circle of radius 2 centered at the origin lying between the $x$-axis and the line $y = x$.

Solution.

(a) (i) In this region we have $-2 \leq x \leq 0$ and $-x \leq y \leq 2$, so the integral is

$$\int_{-2}^{0} \int_{-x}^{2} d\,y \,d\,x.$$ 

(ii) In this region we have $0 \leq y \leq 2$, $-y \leq x \leq 0$, so the integral is

$$\int_{0}^{2} \int_{-y}^{0} d\,x \,d\,y.$$ 

(c) (i) We split the integral, depending on whether $x \leq \sqrt{2}$ or $x \geq \sqrt{2}$. If $0 \leq x \leq \sqrt{2}$ then $0 \leq y \leq x$. If $\sqrt{2} \leq x \leq 2$, then $0 \leq y \leq \sqrt{4-x^2}$. So the integral is

$$\int_{0}^{\sqrt{2}} \int_{0}^{x} d\,y \,d\,x + \int_{\sqrt{2}}^{2} \int_{0}^{\sqrt{4-x^2}} d\,y \,d\,x.$$ 

(ii) In this region we have $0 \leq y \leq \sqrt{2}$ and $y \leq x \leq \sqrt{4-y^2}$, so the integral is

$$\int_{0}^{\sqrt{2}} \int_{y}^{\sqrt{4-y^2}} d\,x \,d\,y.$$ 

3. (3A-3c) Integrate the function $y$ over the triangle with vertices $(\pm 1, 0), (0, 1)$.

*Solution.*

In this region, we have $0 \leq y \leq 1$, $y - 1 \leq x \leq 1 - y$. Hence the integral is

\[
\iint_R y \, dA = \int_0^1 \int_{y-1}^{1-y} y \, dx \, dy = \int_0^1 (2y - 2y^2) \, dx = \left[ y^2 - \frac{2}{3} y^3 \right]_0^1 = \frac{1}{3}.
\]
4. (3A-4) Find the volume of the following via double integration.

(a) The solid lying under the graph of \( z = \sin^2 x \) and over the region bounded below by the \( x \)-axis and above by the central arch of the graph of \( \cos x \).

(c) The solid lying beneath the surface \( z = x^2 - y^2 \), above the \( xy \)-plane, and between the planes \( x = 0 \), \( x = 1 \).

**Solution.**

(a) The volume is

\[
\iint_{R} \sin^2 x \,dA = \int_{-\pi/2}^{\pi/2} \int_{0}^{\cos x} \sin^2 x \,dy \,dx = \int_{-\pi/2}^{\pi/2} \sin^2 x \cos x \,dx = \left[ \frac{1}{3} \sin^3 x \right]_{-\pi/2}^{\pi/2} = \frac{2}{3}.
\]

(c) The function \( x^2 - y^2 \) is zero on the lines \( y = \pm x \) and positive in the region between these two lines lying between \( x = 0 \), \( x = 1 \). Therefore the volume is

\[
\iint_{R} (x^2 - y^2) \,dA = \int_{0}^{1} \int_{-x}^{x} (x^2 - y^2) \,dy \,dx \\
= \int_{0}^{1} \left[ x^2y - \frac{1}{3}y^3 \right]_{-x}^{x} \,dx \\
= \int_{0}^{1} \frac{4}{3} x^3 \,dx \\
= \left[ \frac{1}{3} x^4 \right]_{0}^{1} \\
= \frac{1}{3}.
\]
5. (3A-5) Evaluate the following by changing the order of integration.

(b) \[ \int_0^{1/4} \int_{\sqrt{t}}^{1/2} \frac{e^u}{u} \, du \, dt \]

(c) \[ \int_0^1 \int_{x/3}^1 \frac{1}{1 + u^4} \, du \, dx \]

**Solution.**

(b) We have

\[
\int_0^{1/4} \int_{\sqrt{t}}^{1/2} \frac{e^u}{u} \, du \, dt = \int_0^{1/2} \int_0^{e^{u^2}} \frac{e^u}{u} \, du \, dt \\
= \int_0^{1/2} e^{u^2} \, du \\
= [(u - 1)e^{u^2}]_0^{1/2} \\
= 1 - \frac{1}{2}e.
\]

(c) We have

\[
\int_0^1 \int_{x/3}^1 \frac{1}{1 + u^4} \, du \, dx = \int_0^1 \int_0^{u^3} \frac{1}{1 + u^4} \, dx \, du \\
= \int_0^1 \frac{u^3}{1 + u^4} \, du \\
= \left[ \frac{1}{4} \ln(1 + u^3) \right]_0^1 \\
= \frac{1}{4} \ln 2.
\]
6. Find the volumes above the $xy$-plane bounded by the given surfaces.

(a) The paraboloid $z = x^2 + y^2$ and the planes $x = \pm 1, y = \pm 1$.

(b) The cylinder $y = 4 - x^2$ and the planes $y = 3x$, $z = x + 4$.

Solution.

(a) This volume will be

$$
\int_{-1}^{1} \int_{-1}^{1} (x^2 + y^2) \, dy \, dx = \int_{-1}^{1} \left[ x^2 y + \frac{1}{3} y^3 \right]_{y=-1}^{y=1} \, dx
$$

$$
= \int_{-1}^{1} \left( 2x^2 + \frac{2}{3} \right) \, dx
$$

$$
= \left[ \frac{2}{3} x^3 + \frac{2}{3} x \right]_{-1}^{1}
$$

$$
= \frac{8}{3}.
$$

(b) In the $xy$-plane, the graphs of $y = 4 - x^2$ and $y = 3x$ intersect at the points $(1, 3)$ and $(-4, -12)$. Hence the bounds for $x$ are $-4 \leq x \leq 1$. This volume will be

$$
\int_{-4}^{1} \int_{3x}^{4-x^2} (x + 4) \, dy \, dx = \int_{-4}^{1} (x + 4)(4 - 3x - x^2) \, dx
$$

$$
= \int_{-4}^{1} (16 - 8x - 7x^2 - x^3) \, dx
$$

$$
= \left[ 16x - 4x^2 - \frac{7}{3} x^3 - \frac{1}{4} x^4 \right]_{-4}^{1}
$$

$$
= \frac{625}{12}.
$$