1 Lecture review

1.1 Chain rule

<table>
<thead>
<tr>
<th>Single variable chain rule</th>
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<tr>
<td>If $y = f(g(t))$, then $\frac{dy}{dt} = f'(g(t))g'(t)$</td>
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The analogue of the right hand side is the easiest to write down for multivariable functions:

<table>
<thead>
<tr>
<th>Some examples of multivariable chain rule</th>
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<tbody>
<tr>
<td>If $w = w(x, y)$, $x = x(t)$, $y = y(t)$, then $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$</td>
</tr>
<tr>
<td>If $w = w(x, y)$, $x = x(u, v)$, $y = y(u, v)$, then $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u}$</td>
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<tr>
<td>$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v}$</td>
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The chain rule also helps us find derivatives when functions are defined implicitly. For example, if $f(x, y, z) = 0$ and one wants to compute $\frac{\partial z}{\partial x}$, then take $\frac{\partial}{\partial x}$ of both sides and solve for $\frac{\partial z}{\partial x}$. 
2 Problems

1. (2E-1) In the following, find $dw/ dt$ for the composite function $w = f(x(t), y(t), z(t))$ in two ways:

- use the chain rule, then express your answer in terms of $t$ by using $x = x(t)$, etc.
- express the composite function $f$ in terms of $t$, and then differentiate.

(a) $w = xyz$, $x = t$, $y = t^2$, $z = t^3$

(b) $w = x^2 - y^2$, $x = \cos t$, $y = \sin t$.

2. (2E-2) In each of these, information about the gradient of an unknown function $f(x, y)$ is given; $x$ and $y$ are in turn functions of $t$. Use the chain rule to find out additional information about the composite function $w = f(x(t), y(t))$, without trying to determine $f$ explicitly.

(c) $\nabla f = (1, -1, 2)$ at $(1, 1, 1)$. Let $x = t$, $y = t^2$, $z = t^3$; find $df/ dt$ at $t = 1$.

(d) $\nabla f = (3x^2y, x^3 + z, y)$; $x = t$, $y = t^2$, $z = t^3$. Find $df/ dt$.

3. (2E-4) Let $w = f(x, y)$, and assume that $\nabla w = 2i + 3j$ at $(0, 1)$. If $x = u^2 - v^2$, $y = uv$, find $\partial w/ \partial u$, $\partial w/ \partial v$ at $(u, v) = (1, 1)$.

4. (2E-6) Let $w = f(x, y)$ and make the change of variables $x = u^2 - v^2$, $y = 2uv$. Show that

\[
(w_x)^2 + (w_y)^2 = \frac{(w_u)^2 + (w_v)^2}{4(u^2 + v^2)}.
\]

5. Compute $\partial z/ \partial x$ and $\partial z/ \partial y$ for the following.

(a) $\ln z = z + 2y - 3x$.

(b) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = 9$.

(c) $z = xyz$.$\sin xz$.

(d) $\sin xy + \sin yz + \sin xz = 1$.

3 Answers

1. (a) $6t^5$, (b) $-2\sin 2t = -4\sin t \cos t$.

2. (c) 5, (d) $10t^4$.

3. (a) $\frac{\partial x}{\partial u} = 7$, $\frac{\partial x}{\partial v} = -1$.

4. -

5. (a) $\frac{\partial z}{\partial x} = \frac{3z}{z - 1}$, $\frac{\partial z}{\partial x} = -\frac{2z}{z - 1}$.

5. (b) $\frac{\partial z}{\partial x} = \frac{1 + z^2}{1 + x^2}$, $\frac{\partial z}{\partial y} = -\frac{1 + z^2}{1 + y^2}$.

5. (c) $\frac{\partial z}{\partial x} = \frac{xyz \cos xz + y \sin xz}{1 - x^2 y \cos xz}$, $\frac{\partial z}{\partial y} = \frac{x \sin xz}{1 - x^2 y \cos xz}$.

5. (d) $\frac{\partial z}{\partial x} = -\frac{y \cos xy + z \cos xz}{x \cos xz + y \cos yz}$, $\frac{\partial z}{\partial y} = -\frac{x \cos xy + z \cos yz}{x \cos xz + y \cos yz}$. 