1 Lecture review

1.1 Improper integrals

1. An improper integral is one such that (i) its upper bound is $\infty$, or (ii) its lower bound is $-\infty$, or (iii) both. We have

$$\int_a^\infty f(x) \, dx = \lim_{R \to \infty} \int_a^R f(x) \, dx.$$ 

2. (Comparison test.) If $|f(x)| \leq g(x)$ for all $a \leq x \leq b$, then

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b g(x) \, dx.$$ 

Therefore, if $|f(x)| \leq g(x)$ for all $x \geq a$, then

$$\left| \int_a^\infty f(x) \, dx \right| \leq \int_a^\infty g(x) \, dx.$$ 

3. (Limit comparison test.) Suppose $f(x)$ and $g(x)$ are nonnegative functions. If the following limit

$$L = \lim_{x \to \infty} \frac{f(x)}{g(x)}$$

exists, then (assuming $f$ and $g$ do not behave badly at $a$):

(a) if $L = 0$ and $\int_a^\infty g(x) \, dx$ converges, then so does $\int_a^\infty f(x) \, dx$.

(b) if $0 < L < \infty$, then $\int_a^\infty f(x) \, dx$, $\int_a^\infty g(x) \, dx$ either both converge or both diverge.

(c) if $L = \infty$ and $\int_a^\infty g(x) \, dx$ diverges, then so does $\int_a^\infty f(x) \, dx$. 
2 Problems

1. Compute the following integrals.

(a) \[ \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} \]

(b) \[ \int_{-\infty}^{0} e^x (\cos x - \sin x) \, dx \]

(c) \[ \int_{0}^{\infty} \frac{\sin x - x \cos x}{x^2} \, dx \] (Hint: integrate by parts)

Solution.

(a) We have
\[ \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = [\arctan x]_{-\infty}^{\infty} = \pi. \]

(b) We have
\[ \int_{-\infty}^{0} e^x (\cos x - \sin x) \, dx = [e^x \cos x]_{-\infty}^{0} = 1. \]

(c) We have
\[ \int_{0}^{\infty} \frac{x \cos x - \sin x}{x^2} \, dx = \left[ -\frac{\sin x}{x} \right]_{0}^{\infty} = 1. \]
2. Determine whether the following integrals converge or diverge. If it diverges, determine whether the integral goes off to ±∞ or if it does not exist.

(a) \( \int_0^\infty \frac{\sin x}{e^x} \, dx \)

(b) \( \int_{-\infty}^0 \frac{e^x}{x} \, dx \)

(c) \( \int_{-\infty}^\infty \frac{dx}{(x^4 + 1)^{1/3}} \)

(d) \( \int_0^\infty \sin x \, dx \)

**Solution.**

(a) Since \( \left| \frac{\sin x}{e^x} \right| \leq e^{-x} \) and \( \int_0^\infty e^{-x} \, dx = 1 < \infty \) it follows that this integral converges.

(b) Since the integrand is always negative and it is less than \( e^{-1/x} \) in the range \([-1, 0]\), it follows that this integral diverges to \(-\infty\).

(c) The integral is symmetric so it suffices to consider

\[ 2 \int_0^\infty \frac{dx}{(x^4 + 1)^{1/3}}. \]

Since the integrand is always bounded above by 1, we may as well consider

\[ 2 \int_1^\infty \frac{dx}{(x^4 + 1)^{1/3}}. \]

Now a comparison with

\[ 2 \int_1^\infty \frac{dx}{x^{4/3}} = 6 < \infty \]

shows that the original integral converges.

(d) This integral diverges since the limit

\[ \lim_{R \to \infty} (1 - \cos R) \]

does not exist.