1 Lecture review

1.1 Inverse substitution and trigonometric integrals

1. Trigonometric substitution is a common way to evaluate integrals.

   (a) If you see $\sqrt{a^2 - x^2}$ then consider substituting $x(t) = a \sin t$ or $x = a \cos t$.

   (b) If you see $\sqrt{a^2 + x^2}$ or $a^2 + x^2$ then consider substituting $x(t) = a \tan t$.

2. When evaluating the trigonometric integral $\int \sin^a x \cos^b x \, dx$...

   i. If $a$ is odd, substitute $u = \cos x$.

   ii. If $b$ is odd, substitute $u = \sin x$.

   iii. If $a$ and $b$ are even, use the double angle formula to substitute $\sin x \cos x = \frac{1}{2} \sin 2x$ and use either (i) or (ii)

      (If $a$ and $b$ are both odd, then you can pick either (i) or (ii) to use.)

The following identities are helpful in those cases.

$$\sin^2 x + \cos^2 x = 1$$
$$\sin(2x) = 2 \sin x \cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$
$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$
$$1 + \tan^2 x = \sec^2 x$$
2 Problems

1. Evaluate the following integrals.

(a) \( \int \sin^8 x \cos x \, dx \)

(b) \( \int \sin^4 x \, dx \)

(c) \( \int \frac{\sin^3 x}{\cos^5 x} \, dx \)

Solution.

(a) Substituting \( u = \sin x \) gives \( du = \cos x \, dx \) and hence the integral becomes

\[
\int \sin^8 x \cos x \, dx = \int u^8 \, du = \frac{1}{9}u^9 + C = \frac{1}{9} \sin^9 x + C.
\]

(b) Since

\[
\sin^4 x = \left(\frac{1 - \cos 2x}{2}\right)^2 = \frac{1 - 2 \cos 2x + \cos^2 2x}{4} = \frac{1 - 2 \cos 2x + \frac{1 + \cos 4x}{2}}{4},
\]

it follows that

\[
\int \sin^4 x \, dx = \frac{1}{8} \int (3 - 4 \cos 2x + \cos 4x) \, dx = \frac{1}{32} (12x - 8 \sin 2x + \sin 4x) + C.
\]

(c) Substituting \( u = \cos x \) gives \( du = -\sin x \, dx \) and hence

\[
\int \frac{\sin^3 x}{\cos^5 x} \, dx = \int -\frac{(1 - u^2)}{u^5} \, du = \frac{1}{4} u^{-4} - \frac{1}{2} u^{-2} + C = \frac{1}{4} \sec^4 x - \frac{1}{2} \sec^2 x + C.
\]

This can also be expressed as

\[
\frac{1}{4} \sec^4 x - \frac{1}{2} \sec^2 x + C = \frac{1}{4} \tan^4 x + \left(C - \frac{1}{4}\right).
\]
2. (Supplementary notes, 5C-13) Find the length of the curve \( y = \ln \sin x \) for \( \pi/4 \leq x \leq \pi/2 \).

Solution.
The arc length is

\[
\int_{\pi/4}^{\pi/2} \sqrt{1 + \left( \frac{d}{dx} (\ln \sin x) \right)^2} \, dx = \int_{\pi/4}^{\pi/2} \csc x \, dx = [- \ln(\csc x + \cot x)]_{\pi/4}^{\pi/2} = \ln(1 + \sqrt{2}).
\]
3. Evaluate the following integrals.

(a) \[ \int \frac{x^2 \, dx}{\sqrt{1 - x^2}} \]

(b) \[ \int \frac{\sqrt{1 - x^2}}{x^2} \, dx \]

(c) \[ \int \frac{dx}{(4 + x^2)^{3/2}} \]

(d) \[ \int \frac{dx}{(1 + x^2)^2} \]

Solution.

(a) Substituting \( x = \sin t \) gives \( dx = \cos t \, dt \) and hence the integral is

\[
\int \frac{x^2 \, dx}{\sqrt{1 - x^2}} = \int \sin^2 t \, dt = \frac{1}{2} \left( x - \frac{1}{2} \sin 2x \right) + C = \frac{1}{2} (x - \sin x \cos x) + C = \frac{1}{2} \left( \sin^{-1} x - x \sqrt{1 - x^2} \right) + C.
\]

(b) Substituting \( x = \sin t \) gives \( dx = \cos t \, dt \) and hence the integral is

\[
\int \frac{\sqrt{1 - x^2}}{x^2} \, dx = \int \cot^2 t \, dt = \int (\csc^2 t - 1) \, dt = (-t - \cot t) + C = - \sin^{-1} x - \frac{\sqrt{1 - x^2}}{x} + C.
\]

(c) Substituting \( x = 2 \tan t \) gives \( dx = 2 \sec^2 t \, dt \) and hence the integral is

\[
\int \frac{dx}{(4 + x^2)^{3/2}} = \int \frac{2 \sec^2 t \, dt}{8 \sec^3 t} = \frac{1}{4} \sin t + C = \frac{x}{4 \sqrt{x^2 + 4}} + C.
\]

(d) Substituting \( x = \tan t \) gives \( dx = \sec^2 t \, dt \) and hence the integral is

\[
\int \frac{dx}{(1 + x^2)^2} = \int \cos^2 t \, dt = \frac{1}{2} (t + \sin t \cos t) + C = \frac{1}{2} \left( \tan^{-1} x + \frac{x}{x^2 + 1} \right) + C.
\]