1 Lecture review

1.1 Applications of Integration

1. (Surface area) The surface area of the solid of revolution given by rotating the area beneath the portion of the curve $y = f(x)$ in the interval $[a, b]$ around the $x$-axis is

$$\int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx.$$ 

1.2 Work and average value

1. The work done in moving from $x = a$ to $x = b$ in the presence of a force $F(x)$ is

$$W = \int_a^b F(x) \, dx.$$ 

2. Newton’s law of gravitation says that the force between two masses of $M$ and $m$ at a distance $r$ apart is

$$F = \frac{GMm}{r^2}.$$ 

3. The average value of a function $f(x)$ over an interval $[a, b]$ is

$$\frac{1}{b-a} \int_a^b f(x) \, dx.$$ 

1.3 Integration by direct substitution

1. Translating the chain rule in terms of a statement about integration gives “integration by direct substitution.” It says that

$$\int_a^b f(u(x))u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du.$$ 

This is commonly referred to as the “$u$-substitution” where $u = u(x)$. 
2 Problems

1. Compute the surface area of the following shapes.

   (a) The shape generated by rotating the graph of \( y = x^3 \) in the interval \([0, 1]\) around the \( x \)-axis.

   (b) The shape generated by rotating the graph of \( y = \frac{1}{4}x^2 \) in the interval \([0, 2\sqrt{3}]\) around the \( y \)-axis.

2. (a) (Supplementary notes, 4D'-3) A heavy-duty rubber firehose hanging over the side of a building is 50 feet long and weighs 2 lb./foot. How much work is done winding it up on a windlass on the top of the building?

   (b) Compute the work done in pumping a spherical tank of radius \( R \) (with its bottom touching the ground) full with water of weight \( w \) per unit volume.

   (c) (Supplementary notes, 4D'-4) Two point-particles having respective masses \( m_1 \) and \( m_2 \) are at \( d \) units distance. How much work is required to move them \( n \) times as far apart (i.e., to distance \( nd \))? What is the work to move them infinitely far apart?

3. (a) Compute the average value of the function \( f(x) = x^2 \) in the interval \([2, 5]\).

   (b) What is the average cross-sectional area of a hemisphere of radius \( R \) given by \( x^2 + y^2 + z^2 \leq R^2 \), \( z \geq 0 \)? (Cross-sections are taken parallel to the \( xy \)-plane.)

   (c) (Supplementary notes, 4D-4) What is the average value of the square of the distance of a point \( P \) from a fixed point \( Q \) on the unit circle, where \( P \) is chosen at random on the circle? (Use coordinates; place \( Q \) on the \( x \)-axis.) Check your answer for reasonableness.

4. Evaluate the following integrals.

   (a) \( \int_0^1 2x \cos(x^2) \, dx \)

   (b) \( \int_e^3 \frac{dx}{x \ln x} \)

   (c) \( \int_0^3 \frac{3x}{\sqrt{x+1}} \, dx \)

3 Answers

1. (a) \( \frac{56\pi}{3} \), (b) \( \frac{56\pi}{3} \).

2. (a) 2500 ft-lbs, (b) \( 8\pi w R^3/3 \), (c) \( \frac{Gm_1m_2}{d} \left( \frac{n-1}{n} \right) \), and as \( n \to \infty \) this is \( \frac{Gm_1m_2}{d} \)

3. (a) 13, (b) \( \frac{2\pi R^3}{3} \), (c) 2

4. (a) \( \sin(1) \), (b) \( \ln \ln 3 \), (c) 8