1  Lecture review

1.1 Applications of Integration

1. (Surface area) The surface area of the solid of revolution given by rotating the area beneath the portion of the curve \( y = f(x) \) in the interval \([a, b]\) around the \( x \)-axis is

\[
\int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx.
\]

1.2 Work and average value

1. The work done in moving from \( x = a \) to \( x = b \) in the presence of a force \( F(x) \) is

\[
W = \int_a^b F(x) \, dx.
\]

2. Newton’s law of gravitation says that the force between two masses of \( M \) and \( m \) at a distance \( r \) apart is

\[
F = \frac{GMm}{r^2}.
\]

3. The average value of a function \( f(x) \) over an interval \([a, b]\) is

\[
\frac{1}{b-a} \int_a^b f(x) \, dx.
\]

1.3 Integration by direct substitution

1. Translating the chain rule in terms of a statement about integration gives “integration by direct substitution.” It says that

\[
\int_a^b f(u(x))u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du.
\]

This is commonly referred to as the “u-substitution” where \( u = u(x) \).
2 Problems

1. Compute the surface area of the following shapes.

   (a) The shape generated by rotating the graph of \( y = x^3 \) in the interval [0, 1] around the \( x \)-axis.

   (b) The shape generated by rotating the graph of \( y = \frac{1}{4}x^2 \) in the interval [0, 2\( \sqrt{3} \)] around the \( y \)-axis.

**Solution.**

(a) Since \( y'(x) = 3x^2 \), the surface area is

\[
\int_0^1 2\pi(x^3)\sqrt{1+(3x^2)^2} \, dx = \frac{\pi}{27} [(1 + 9x^4)^{3/2}]_0^1 = \frac{\pi}{27} (10\sqrt{10} - 1).
\]

(b) Since \( x(y) = 2\sqrt{y} \) we have that \( x'(y) = y^{-1/2} \). The range of \( y \) is 0 ≤ \( y \) ≤ 3. Hence the surface area is

\[
\int_0^3 2\pi(2\sqrt{y})\sqrt{1+(y^{-1/2})^2} \, dy = 4\pi \int_0^3 \sqrt{y+1} \, dy = 4\pi \left[ \frac{2}{3}(y+1)^{3/2} \right]_0^3 = \frac{56\pi}{3}.
\]
2. (a) (Supplementary notes, 4D’-3) A heavy-duty rubber firehose hanging over the side of a building is 50 feet long and weighs 2 lb./foot. How much work is done winding it up on a windlass on the top of the building?

(b) Compute the work done in pumping a spherical tank of radius $R$ (with its bottom touching the ground) full with water of weight $w$ per unit volume.

(c) (Supplementary notes, 4D’-4) Two point-particles having respective masses $m_1$ and $m_2$ are at $d$ units distance. How much work is required to move them $n$ times as far apart (i.e., to distance $nd$)? What is the work to move them infinitely far apart?

**Solution.**

(a) The work element is $2h \, dh$ and hence the total work done is

$$\int_0^{50} 2h \, dh = 2500 \text{ft-lbs.}$$

(b) The work element is

$$dW = w\pi(R^2 - (R - h)^2)(h) \, dh = w\pi h^2(2R - h) \, dh$$

and hence the total work is

$$\int_0^{2R} dW = w\pi \int_0^{2R} h^2(2R - h) \, dh = \frac{8\pi wR^3}{3}.$$

(c) If they are $x$ units apart, then the gravitational force between them is $\frac{Gm_1m_2}{x^2}$ and hence the work is

$$\int_d^{nd} \frac{Gm_1m_2}{x^2} \, dx = \frac{Gm_1m_2}{d} \left( \frac{n - 1}{n} \right).$$

As $n \to +\infty$, this approaches $\frac{Gm_1m_2}{d}$. 
3. (a) Compute the average value of the function \( f(x) = x^2 \) in the interval \([2, 5]\).

(b) What is the average cross-sectional area of a hemisphere of radius \( R \) given by 
\[ x^2 + y^2 + z^2 \leq R^2, \quad z \geq 0 \] (Cross-sections are taken parallel to the \( xy \)-plane.)

(c) (Supplementary notes, 4D-4) What is the average value of the square of the distance of a point \( P \) from a fixed point \( Q \) on the unit circle, where \( P \) is chosen at random on the circle? (Use coordinates; place \( Q \) on the \( x \)-axis.) Check your answer for reasonableness.

**Solution.**

(a) The average value is
\[
\frac{1}{3} \int_2^5 x^2 \, dx = \frac{1}{3} \left( \frac{5^3 - 2^3}{3} \right) = 13.
\]

(b) The cross-sectional area at height \( z \) is \( \pi(R^2 - z^2) \) and hence the average is
\[
\frac{1}{R} \int_0^R \pi(R^2 - z^2) \, dz = \frac{2\pi R^2}{3}.
\]

(c) By symmetry, restrict \( P \) to the upper semicircle. By the law of cosines, we see that \( |PQ|^2 = 2 - 2\cos \theta \) and hence
\[
\text{average value of } |PQ|^2 = \frac{1}{\pi} \int_0^\pi (2 - 2\cos \theta) \, d\theta = 2.
\]
4. Evaluate the following integrals.

(a) \[ \int_0^1 2x \cos(x^2) \, dx \]

(b) \[ \int_{\ln e}^3 \frac{dx}{x \ln x} \]

(c) \[ \int_0^3 \frac{3x}{\sqrt{x+1}} \, dx \]

Solution.

(a) Substituting \( u = x^2 \) gives \( du = 2x \, dx \) and hence the integral becomes

\[ \int_{x=0}^{x=1} 2x \cos(x^2) \, dx = \int_{u=0}^{u=1} \cos u \, du = \sin(1). \]

(b) Substituting \( u = \ln x \) gives \( du = \frac{1}{x} \, dx \) and hence the integral becomes

\[ \int_{x=e}^{x=3} \frac{dx}{x \ln x} = \int_{u=1}^{u=\ln 3} \frac{du}{u} = \ln \ln 3. \]

(c) Substituting \( u = x + 1 \) gives \( du = dx \) and hence the integral becomes

\[ \int_{x=0}^{x=3} \frac{3x}{\sqrt{x+1}} \, dx = \int_{u=1}^{u=4} \frac{3(u-1)}{\sqrt{u}} \, du = \left[ 2u^{3/2} - 6u^{1/2} \right]_1^4 = 8. \]