1 Lecture review

1.1 Integration

1. The definite integral $\int_{a}^{b} f(x) \, dx$ computes the area under the graph of $y = f(x)$ over the interval $a \leq x \leq b$.

2. The Riemann sum approximates this area by subdividing into $n$ small rectangles. The endpoints of the $i$th rectangle is $[x_{i-1}, x_i]$, where $x_i = a + \left( \frac{b-a}{n} \right) i$. By choosing a point $c_i$ in $[x_{i-1}, x_i]$, the total area is approximated by

$$\sum_{i=1}^{n} f(c_i) \left( \frac{b-a}{n} \right).$$

(a) If $c_i = x_{i-1}$, then this gives the left Riemann sum.

(b) If $c_i = x_i$, then this gives the right Riemann sum.

(c) If $c_i$ is the point in $[x_{i-1}, x_i]$ where $f$ is minimal, then this gives the lower Riemann sum.

(d) If $c_i$ is the point in $[x_{i-1}, x_i]$ where $f$ is maximal, then this gives the upper Riemann sum.

3. As $n$ grows to infinity, each of the previous Riemann sums converges to $\int_{a}^{b} f(x) \, dx$.

1.2 The First Fundamental Theorem of Calculus

1. If $f(x)$ is continuous, then the antiderivative $F(x)$ exists. It satisfies $F'(x) = f(x)$ and

$$\int_{a}^{b} f(x) \, dx = F(b) - F(a).$$

2. This allows one to compute area under functions without using Riemann sums.
2 Problems

1. Evaluate the following integrals using both of the following methods: (i) right Riemann sum approximation, (ii) FFTC. The following formulas may be helpful.

\[ \sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} \sin(kx) = \frac{\sin \left( \frac{1}{2} nx \right) \sin \left( \frac{1}{2} (n+1)x \right)}{\sin \left( \frac{1}{2} x \right)} \]  

(a) \[ \int_{0}^{5} 5 \, dx \]

(b) \[ \int_{-1}^{1} 2 + 3x \, dx \]

(c) \[ \int_{0}^{1} 4x \, dx \]

(d) \[ \int_{1}^{2} 2 - 4x + 2x^2 \, dx \]

(e) \[ \int_{0}^{1} \sin(x) \, dx \]

Solution.

(a) (i) The right Riemann sum approximation of \[ \int_{0}^{2} 5 \, dx \] with \(2n\) intervals of length \(\frac{1}{n}\) is given by

\[ \frac{1}{n} \sum_{k=1}^{2n} 5 = 10. \]

Thus

\[ \int_{0}^{2} 5 \, dx = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{2n} 5 = 10. \]

(ii) By FFTC

\[ \int_{0}^{2} 5 \, dx = 5x|_{x=0}^{2} = 10. \]

(b) (i) The right Riemann sum approximation of \[ \int_{-1}^{1} 2 + 3x \, dx \] with \(2n\) intervals of length \(\frac{1}{n}\) is given by

\[ \frac{1}{n} \sum_{k=1}^{2n} \left( 2 + 3 \left( -1 + \frac{k}{n} \right) \right) = -2 + \frac{3}{n^2} \sum_{k=1}^{2n} k = -2 + \frac{3}{n^2} \cdot \frac{2n(2n+1)}{2} \]

where the last line uses (1). Thus

\[ \int_{-1}^{1} 2 + 3x \, dx = \lim_{n \to \infty} \left( -2 + \frac{3n(2n+1)}{n^2} \right) = -2 + \lim_{n \to \infty} 3 \left( 2 + \frac{1}{n} \right) = 4. \]
(ii) By FFTC
\[ \int_{-1}^{1} 2 + 3x \, dx = 2x + \frac{3}{2} x^2 \bigg|_{x=-1}^{1} = 4. \]

(c) (i) The right Riemann sum approximation of \( \int_{0}^{1} 4^x \, dx \) with \( n \) intervals of length \( \frac{1}{n} \) is given by
\[ \frac{1}{n} \sum_{k=1}^{n} 4^{k/n} = 4^{1/n} \cdot \frac{1}{n} \cdot 4 - 1 \cdot 4^{1/n} - 1. \]
Thus
\[ \int_{0}^{1} 4^x \, dx = \lim_{n \to \infty} \left( 4^{1/n} \cdot \frac{1}{n} \cdot 4 - 1 \cdot 4^{1/n} - 1 \right) = \lim_{n \to \infty} 4^{1/n} \cdot \lim_{n \to \infty} \left( \frac{1}{n} \cdot \frac{3}{4^{1/n} - 1} \right). \]
Since \( \lim_{n \to \infty} 4^{1/n} = 1 \), we have by L'Hôpital's rule
\[ \int_{0}^{1} 4^x \, dx = \lim_{n \to \infty} \left( \frac{3y}{4y - 1} \right) = \lim_{y \to 0} \frac{3}{4y \ln(4)} = \frac{3}{\ln(4)}. \]

(ii) By FFTC
\[ \int_{0}^{1} 4^x \, dx = \frac{4^x}{\ln(4)} \bigg|_{x=0}^{1} = \frac{3}{\ln(4)}. \]

(d) (i) The right Riemann sum approximation of \( \int_{1}^{2} 2 - 4x + 2x^2 \, dx \) with \( n \) intervals of length \( \frac{1}{n} \) is given by
\[ \frac{1}{n} \sum_{k=1}^{n} \left( 2 - 4 \left( 1 + \frac{k}{n} \right) + 2 \left( 1 + \frac{k}{n} \right)^2 \right) = \frac{2}{n^3} \sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{3n^3} \]
where the second equality uses (1). Thus
\[ \int_{1}^{2} 2 - 4x + 2x^2 \, dx = \lim_{n \to \infty} \frac{n(n + 1)(2n + 1)}{3n^2} = \lim_{n \to \infty} \left( \frac{2}{3} \left(1 + \frac{1}{n}\right) \left(1 + \frac{1}{2n}\right) \right) = \frac{2}{3}. \]

(ii) By FFTC
\[ \int_{1}^{2} 2 - 4x + 2x^2 \, dx = 2 \int_{1}^{2} (x - 1)^2 \, dx = \frac{2}{3} (x - 1)^3 \bigg|_{1}^{2} = \frac{2}{3}. \]

(e) (i) The right Riemann sum approximation of \( \int_{0}^{1} \sin x \, dx \) with \( n \) intervals of length \( \frac{1}{n} \) is given by
\[ \frac{1}{n} \sum_{k=1}^{n} \sin \left( \frac{k}{n} \right) = \frac{1}{n} \cdot \frac{\sin \left( \frac{1}{2} \right) \sin \left( \frac{1}{2} + \frac{1}{2n} \right)}{\sin \left( \frac{1}{2n} \right)}. \]
where we used (1). Then
\[
\int_0^1 \sin(x) \, dx = \sin(1/2) \lim_{n \to \infty} \sin \left( \frac{1}{2} + \frac{1}{2n} \right) \cdot \lim_{n \to \infty} \frac{1}{n \sin \left( \frac{1}{2n} \right)} = \sin(1/2)^2 \lim_{y \to 0} \frac{y}{\sin(y/2)}.
\]

By L’Hôpital’s rule, we get
\[
\int_0^1 \sin(x) \, dx = \sin(1/2)^2 \lim_{y \to 0} \frac{1}{\cos(y/2)/2} = 2 \sin(1/2)^2.
\]

Recall that \(1 - 2 \sin(\theta/2)^2 = \cos(\theta)\), thus
\[
\int_0^1 \sin(x) \, dx = 1 - \cos(1).
\]

(ii) By FFTC
\[
\int_0^1 \sin(x) \, dx = -\cos(x) \bigg|_{x=0}^1 = 1 - \cos(1).
\]
2. Compute the following limits by finding a Riemann sum and using FFTC.

(a) \( \lim_{n \to \infty} \left( \frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \frac{n}{n^2 + 3^2} + \cdots + \frac{n}{n^2 + (n-1)^2} + \frac{n}{n^2 + n^2} \right) \)

(b) \( \lim_{n \to \infty} \left( \left(1 + \frac{1}{n}\right)^{1/n} \left(1 + \frac{2}{n}\right)^{1/n} \cdots \left(1 + \frac{n-2}{n}\right)^{1/n} \left(1 + \frac{n-1}{n}\right)^{1/n} \right) \)

Solution.

(a) We can write

\[
\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \cdots + \frac{n}{n^2 + n^2} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + \frac{k^2}{n^2}}.
\]

This is a right Riemann sum for the definite integral

\[
\int_0^1 \frac{1}{1 + x^2} \, dx = \arctan(x)|_{x=0}^{x=1} = \frac{\pi}{4}.
\]

Therefore

\[
\lim_{n \to \infty} \left( \frac{n}{n^2 + 1} + \frac{n}{n^2 + 2^2} + \cdots + \frac{n}{n^2 + n^2} \right) = \frac{\pi}{4}.
\]

(b) Notice that

\[
\ln \left( \left(1 + \frac{1}{n}\right)^{1/n} \left(1 + \frac{2}{n}\right)^{1/n} \cdots \left(1 + \frac{n-1}{n}\right)^{1/n} \right) = \frac{1}{n} \sum_{k=0}^{n-1} \ln \left(1 + \frac{k}{n}\right)
\]

which is a left Riemann sum for the definite integral

\[
\int_1^2 \ln(x) \, dx = x \ln(x) - x|_{x=1}^{x=2} = 2 \ln(2) - 1 = \ln(4) - 1.
\]

Then

\[
\lim_{n \to \infty} \left( \left(1 + \frac{1}{n}\right)^{1/n} \left(1 + \frac{2}{n}\right)^{1/n} \cdots \left(1 + \frac{n-1}{n}\right)^{1/n} \right) = e^{\ln(4)-1} = \frac{4}{e}.
\]