1 Lecture review

1.1 Linear and quadratic approximations

1. The slope of $y = f(x)$ at $x = x_0$ is $f'(x_0)$.

2. The tangent line at $(x_0, y_0)$ for $y = f(x)$ has equation

$$y = f(x_0) + f'(x_0)(x - x_0)$$

This tangent line is referred to the linear approximation of $y = f(x)$ at $x = x_0$. Of all linear functions (lines), this tangent line is the one that best approximates $y = f(x)$ near $x = x_0$.

3. The quadratic approximation at $(x_0, y_0)$ for $y = f(x)$ is

$$y = f(x_0) + f'(x_0)(x - x_0) + rac{f''(x_0)}{2}(x - x_0)^2.$$

Of all quadratic functions (parabolas), this one is the best approximates $y = f(x)$ near $x = x_0$.

4. Using the “building blocks” of quadratic approximation (near $x = 0$)

$$e^x \approx 1 + x + \frac{x^2}{2}$$

$$\sin x \approx x$$

$$\cos x \approx 1 - \frac{x^2}{2}$$

$$(1 + x)^k \approx 1 + kx + \frac{k(k - 1)}{2}x^2$$

$$\ln(1 + x) \approx x - \frac{x^2}{2}$$

one can compute quadratic approximations of a large class of functions (without needing to take derivatives.)

1.2 L’Hôpital’s rule

1. If $f(x)$ is continuous at $x = a$, then $\lim_{x \to a} f(x) = f(a)$. (No need for L’Hôpital’s rule.)

2. If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form and $f'(x)$, $g'(x)$ are defined at $x = a$ with $g'(x) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided that the limit on the right hand side exists.
2 Problems

1. Find the best quadratic approximation to the following functions at the specified points in two ways: (i) by taking derivatives and using the direct formula, and (ii) without taking derivatives and instead using the quadratic approximation building blocks.

   (a) \( y = e^{x^2}, \ x = 0 \)
   (b) \( y = e^x \cos(x), \ x = 0 \)
   (c) \( y = \sqrt{x}, \ x = 4 \)
   (d) \( y = \frac{x+1}{x^2 - 3x^2 + 3x}, \ x = 1 \)
   (e) \( y = e^{3x^2} (\cos(3x))^{2/3}, \ x = 0 \)

2. Evaluate the following limits.

   (a) \( \lim_{x \to 1} \frac{xe^{3x}}{\sin(2x)} \)
   (b) \( \lim_{x \to 0} \frac{\sin(x^2)}{1 - \cos x} \)
   (c) \( \lim_{x \to 1} \frac{\ln x}{(x-1)^3} \)
   (d) \( \lim_{x \to \infty} \left( \sqrt{x^2 + x} - \sqrt[3]{x^3 + x^2} \right) \)
   (e) \( \lim_{x \to \infty} \left( \cos \left( \frac{1}{x} \right) \right)^x \)
   (f) (Hard.) \( \lim_{x \to 0} \frac{e^{x^2} + 2 \cos x - 3}{\sin^2(1 - \cos x)} \)
   (g) (Even harder.) \( \lim_{x \to 0} \frac{\ln^3(\sqrt{1 + 2x} - \sin x)}{\sin^2(x - \sin x)} \)

3 Answers

1. (a) 1 + \( x^2 \), (b) 1 + \( x \), (c) 2 + \( \frac{1}{4} \)(\( x - 4 \)) - \( \frac{1}{81} \)(\( x - 4 \))^2, (d) 2 + (\( x - 1 \)), (e) 1
2. (a) 1/2, (b) 2, (c) \( \infty \), (d) 1/6, (e) \( e^{-1/2} \), (f) 7/3, (g) -9/2