1 Lecture review

1.1 Linear and quadratic approximations

1. The slope of \( y = f(x) \) at \( x = x_0 \) is \( f'(x_0) \).

2. The tangent line at \((x_0, y_0)\) for \( y = f(x) \) has equation

\[
y = f(x_0) + f'(x_0)(x - x_0)
\]

This tangent line is referred to the linear approximation of \( y = f(x) \) at \( x = x_0 \). Of all linear functions (lines), this tangent line is the one that best approximates \( y = f(x) \) near \( x = x_0 \).

3. The quadratic approximation at \((x_0, y_0)\) for \( y = f(x) \) is

\[
y = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2.
\]

Of all quadratic functions (parabolas), this one is the best approximates \( y = f(x) \) near \( x = x_0 \).

4. Using the “building blocks” of quadratic approximation (near \( x = 0 \))

\[
e^x \approx 1 + x + \frac{x^2}{2}
\]

\[
\sin x \approx x
\]

\[
\cos x \approx 1 - \frac{x^2}{2}
\]

\[
(1 + x)^k \approx 1 + kx + \frac{k(k - 1)}{2}x^2
\]

\[
\ln(1 + x) \approx x - \frac{x^2}{2}
\]

one can compute quadratic approximations of a large class of functions (without needing to take derivatives.)

1.2 L’Hôpital’s rule

1. If \( f(x) \) is continuous at \( x = a \), then \( \lim_{x \to a} f(x) = f(a) \). (No need for L’Hôpital’s rule.)

2. If \( \lim_{x \to a} \frac{f(x)}{g(x)} \) is of \( \frac{0}{0} \) form or \( \frac{\infty}{\infty} \) form and \( f'(x), g'(x) \) are defined at \( x = a \) with \( g'(x) \neq 0 \), then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},
\]

provided that the limit on the right hand side exists.
2 Problems

1. Find the best quadratic approximation to the following functions at the specified points in two ways: (i) by taking derivatives and using the direct formula, and (ii) without taking derivatives and instead using the quadratic approximation building blocks.

(a) \( y = e^{x^2}, \ x = 0 \)
(b) \( y = e^x \cos(x), \ x = 0 \)
(c) \( y = \sqrt{x}, \ x = 4 \)
(d) \( y = \frac{x+1}{x^2-3x^3+3x}, \ x = 1 \)
(e) \( y = e^{3x^2}(\cos(3x))^{2/3}, \ x = 0 \)

Solution.

(a) (i) The derivatives of \( f(x) = e^{x^2} \) are

\[
\begin{align*}
f'(x) &= 2xe^{x^2} \\
f''(x) &= 2(1+2x^2)e^{x^2}
\end{align*}
\]

from which one obtains \( f'(0) = 0 \) and \( f''(0) = 2 \). Therefore the quadratic approximation is

\[
1 + x^2.
\]

(ii) Since \( e^x \approx 1 + x + \frac{1}{2}x^2 \) is the quadratic approximation for \( e^x \) around zero, we plug in \( x^2 \) now to get \( e^{x^2} \approx 1 + x^2 + \frac{1}{4}x^4 \), which we cut off after the \( x^2 \) term to get the quadratic approximation

\[
1 + x^2.
\]

(b) (i) The derivatives of \( f(x) = e^x \cos x \) are

\[
\begin{align*}
f'(x) &= e^x(\cos x - \sin x) \\
f''(x) &= -2e^x \sin x
\end{align*}
\]

from which one obtains \( f'(0) = 1 \) and \( f''(0) = 0 \). Therefore the quadratic approximation is

\[
1 + x.
\]

(ii) Since \( e^x \approx 1 + x + \frac{1}{2}x^2 \) is the quadratic approximation for \( e^x \) around zero and \( \cos x \approx 1 - \frac{1}{2}x^2 \) approximation for \( \cos x \) around zero, we multiply these and cut off any terms after \( x^2 \) to get

\[
e^x \cos x \approx \left(1 + x + \frac{1}{2}x^2\right) \left(1 - \frac{1}{2}x^2\right) \approx 1 + x.
\]
(c) (i) The derivatives of \( f(x) = \sqrt{x} \) are

\[
f'(x) = \frac{1}{2}x^{-1/2}
\]
\[
f''(x) = -\frac{1}{4}x^{-3/2}
\]

and hence \( f(4) = 2, f'(4) = 1/4, f''(4) = -1/32. Therefore the quadratic approximation is

\[
2 + \frac{1}{4}(x - 4) - \frac{1}{64}(x - 4)^2.
\]

(ii) Rewriting

\[
\sqrt{x} = \sqrt{4 + (x - 4)} = 2 \left( 1 + \frac{x - 4}{4} \right)^{1/2}
\]

and using the quadratic approximation \((1 + x)^{1/2} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 \) gives

\[
2 \left( 1 + \frac{1}{2} \left( \frac{x - 4}{4} \right) - \frac{1}{8} \left( \frac{x - 4}{4} \right)^2 \right) = 2 + \frac{1}{4}(x - 4) - \frac{1}{64}(x - 4)^2.
\]

(d) (i) The derivatives of \( f(x) = \frac{x+1}{x^3-3x^2+3x} \) are

\[
f'(x) = \frac{-2x^3 + 6x - 3}{(x^3 - 3x^2 + 3x)^2}
\]
\[
f''(x) = \frac{6(-1 + x)(-3 + 6x - 6x^2 + x^4)}{(x^3 - 3x^2 + 3x)^3}
\]

from which we see that \( f(1) = 2, f'(1) = 1, f''(1) = 0 \) and hence the quadratic approximation is

\[
2 + (x - 1).
\]

(ii) Rewriting the function \( f(x) = \frac{x+1}{x^3-3x^2+3x} \) as

\[
2 + (x - 1)
\]
\[
\frac{1}{1 + (x - 1)^3}
\]

and then using the quadratic approximation \((1 + x)^{-1} \approx 1 - x + x^2 \) gives

\[
\frac{1}{1 + (x - 1)^3} \approx 1 - (x - 1)^3 - (x - 1)^6. \]

Since we are only interested in the quadratic approximation, we throw away the terms that are higher than quadratic in degree, giving us the quadratic approximation

\[
\frac{1}{1 + (x - 1)^3} \approx 1
\]

and hence

\[
f(x) = \frac{2 + (x - 1)}{1 + (x - 1)^3} \approx 2 + (x - 1).
\]
(e) (i) The derivatives of \( f(x) = e^{3x^2}(\cos(3x))^{2/3} \) are

\[
\begin{align*}
    f'(x) &= 2e^{3x^2}(\cos(3x))^{-1/3}(3x \cos(3x) - \sin(3x)) \\
    f''(x) &= 2e^{3x^2}(\cos 3x)^{-4/3}(18x^2 \cos^2(3x) - \sin^2(3x) - 12x \sin(3x) \cos(3x))
\end{align*}
\]

from which we see that \( f(0) = 1, \ f'(0) = 0, \ f''(0) = 0. \) Therefore the quadratic approximation is just

\[ f(x) \approx 1. \]

(ii) Using the quadratic approximation \( e^x = 1 + x + \frac{x^2}{2} \), we get

\[ e^{3x^2} \approx 1 + 3x^2. \]

From \( \cos x \approx 1 - \frac{1}{2}x^2 \) and \( (1 + x)^{2/3} \approx 1 + \frac{2}{3}x - \frac{1}{9}x^2 \) we get

\[
\cos(3x)^{2/3} \approx \left(1 - \frac{9}{2}x^2\right)^{2/3} \approx 1 - 3x^2
\]

and hence

\[ e^{3x^2} \cos(3x)^{2/3} \approx (1 + 3x^2)(1 - 3x^2) \approx 1. \]
2. Evaluate the following limits.

(a) \( \lim_{x \to 0} \frac{xe^{3x}}{\sin(2x)} \)

(b) \( \lim_{x \to 0} \frac{\sin(x^2)}{1 - \cos x} \)

(c) \( \lim_{x \to 1} \frac{\ln x}{(x - 1)^3} \)

(d) \( \lim_{x \to \infty} \left( \sqrt{x^2 + x} - \sqrt[3]{x^3 + x^2} \right) \)

(e) \( \lim_{x \to \infty} \left( \cos \left( \frac{1}{x} \right) \right)^x \)

(f) (Hard.) \( \lim_{x \to 0} \frac{e^{x^2} + 2 \cos x - 3}{\sin^2(1 - \cos x)} \)

(g) (Even harder.) \( \lim_{x \to 0} \frac{\ln^3(\sqrt{1 + 2x} - \sin x)}{\sin^2(x - \sin x)} \)

**Solution.**

(a) Applying L'Hôpital's rule gives

\[ \lim_{x \to 0} \frac{xe^{3x}}{\sin(2x)} = \lim_{x \to 0} \frac{(1 + 3x)e^{3x}}{2 \cos 2x} = \frac{1}{2}. \]

(b) Applying L'Hôpital's rule twice gives

\[ \lim_{x \to 0} \frac{\sin(x^2)}{1 - \cos x} = \lim_{x \to 0} \frac{2x \cos x^2}{\sin x} = \lim_{x \to 0} \frac{2(\cos x^2 - 2x^2 \sin x^2)}{\cos x} = 2. \]

(c) Applying L'Hôpital's rule gives

\[ \lim_{x \to 1} \frac{\ln x}{(x - 1)^3} = \lim_{x \to 1} \frac{1/x}{3(x - 1)^2} = \infty. \]

(d) Rewriting the function as

\[ \sqrt{x^2 + x} - \sqrt[3]{x^3 + x^2} = (1 + x^{-1})^{1/2} \cdot \frac{(1 + x^{-1})^{1/6} - 1}{x^{-1}}, \]

The first term \((1 + x^{-1})^{-1/2}\) approaches 1 as \(x \to \infty\). Applying L'Hôpital's rule on the second term then gives

\[ \lim_{x \to \infty} \frac{(1 + x^{-1})^{1/6} - 1}{x^{-1}} = \lim_{x \to \infty} \frac{-(1/6)x^{-2}(1 + x^{-1})^{-5/6}}{-x^{-2}} = \frac{1}{6}. \]
(e) Rewrite the function as
\[ \exp \left( x^2 \ln \cos \left( \frac{1}{x} \right) \right), \]
so that we are reduced to computing
\[ \lim_{x \to \infty} x^2 \ln \cos \left( \frac{1}{x} \right) = \lim_{x \to 0} \frac{\ln \cos x}{x^2}. \]

Two applications of L'Hôpital's rule now gives
\[ \lim_{x \to 0} \frac{\ln \cos x}{x^2} = \lim_{x \to 0} \frac{-\tan x}{2x} = \lim_{x \to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}, \]
and hence the original limit is \( e^{-1/2} \).

(f) It would require four applications of L'Hôpital to compute this limit. Instead, we use Taylor series. We have \( e^{x^2} = 1 + x^2 + \frac{1}{2} x^4 + O(x^6) \) and \( 2 \cos x = 2 - x^2 + \frac{1}{12} x^4 + O(x^6) \), so the numerator has Taylor series of the form
\[ e^{x^2} + 2 \cos x - 3 = \frac{7}{12} x^4 + O(x^6) \]
We know that \( 1 - \cos x = \frac{1}{2} x^2 + O(x^4) \) and \( \sin^2 x = x^2 + O(x^4) \), so the denominator has Taylor series of the form
\[ \sin^2(1 - \cos x) = \frac{1}{4} x^4 + O(x^6). \]
and hence the limit is \( (7/12)/(1/4) = 7/3 \).

(g) It would require six applications of L'Hôpital to compute this limit. Instead, we use Taylor series. We know that \( \sqrt{1+2x} - \sin x = (1 + x - \frac{1}{2} x^2) - (x) + O(x^3) = 1 - \frac{1}{2} x^2 + O(x^3) \) and \( \ln(1+x) = x - \frac{1}{2} x^2 + O(x^3) \) which gives \( \ln^3(1-x) = x^3 + O(x^4) \), so the numerator has Taylor series of the form
\[ \ln^3(\sqrt{1+2x} - \sin x) = \ln^3 \left( 1 - \frac{1}{2} x^2 + O(x^3) \right) = -\frac{1}{8} x^6 + O(x^7) \]
Next, we know \( x - \sin x = x^3/6 + O(x^5) \) and \( \sin^2 x = x^2 + O(x^4) \). Hence the denominator has Taylor series of the form
\[ \sin^2(x - \sin x) = \frac{x^6}{36} + O(x^8) \]
and the desired limit is then \( (-1/8)/(1/36) = -9/2 \).