1 Lecture review

1.1 Power series

1. A power series is an expression of the form
\[ \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \]

2. The radius of convergence of a power series is the nonnegative number \( R \) such that the power series converges when \( |x| < R \) and diverges when \( |x| > R \).

3. If \( f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \)
then the derivative and integral of \( f(x) \) may be computed termwise:
\[ f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \]
\[ \int_0^x f(t) \, dt = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots \]

1.2 Taylor series

1. If \( f(x) \) is a nice function, then \( f(x) \) has the power series representation
\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + \frac{1}{6} f^{(3)}(0)x^3 + \frac{1}{24} f^{(4)}(0)x^4 + \cdots \]

2. Cutting off the above series at (i) after the \( x \) term gives the best linear approximation, (ii) after the \( x^2 \) gives the best quadratic approximation, etc.
2 Problems

1. Determine the radius of convergence of the following power series (in $x$).

   (a) $\sum_{n=1}^{\infty} 3^n x^n$
   (b) $\sum_{n=1}^{\infty} n^n x^n$
   (c) $\sum_{n=1}^{\infty} n^3 x^n$
   (d) $\sum_{n=1}^{\infty} \frac{x^n}{(2n)^n}$

2. Compute the Taylor series around the origin for the following functions.

   (a) $f(x) = \frac{x^2}{1-x}$
   (b) $f(x) = \cos(x^2)$
   (c) $f(x) = -\frac{2x}{(1+x^2)^2}$ (Hint: this is the derivative of $\frac{1}{1+x^2}$)
   (d) $f(x) = \arctan(x)$ (Hint: this is the integral of $\frac{1}{1+x^2}$)
   (e) $f(x) = \int_0^x \frac{\sin t}{t} \, dt$
   (f) $f(x) = \int_0^x \sin(e^t - 1) \, dt$ (Up to $x^3$ only.)

3. Find the first three nonzero terms of the solutions to the following ordinary differential equations.

   (a) $f'(x) = 1 + x + f(x)$, $f(0) = 0$
   (b) $f'(x) = x(f(x) - 1)$, $f(0) = 2$
   (c) $f'(x) = 1 - xf(x)^2$, $f(0) = 2$

3 Answers

1. (a) $1/3$, (b) $0$, (c) $1$, (d) $\infty$
2. (a) $\sum_{n=2}^{\infty} x^n$, (b) $\sum_{n=0}^{\infty} (-1)^n x^{4n}/(2n)!$, (c) $\sum_{n=1}^{\infty} (-1)^n (2n)x^{2n-1}$, (d) $\sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n + 1)$
2. (e) $\sum_{n=0}^{\infty} (-1)^n x^{2n+1}/((2n + 1)(2n + 1)!)$, (f) $\frac{1}{2} x^2 + \frac{1}{6} x^3 + \cdots$
3. (a) $x + x^2 + \frac{1}{3} x^3 + \cdots$, (b) $2 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \cdots$, (c) $2 + x - 2x^2 + \cdots$