1 Lecture review

1.1 Infinite series

1. An infinite series is an expression of the form $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots$.

2. The geometric series is $\sum_{i=1}^{\infty} r^i$. If $|r| < 1$, then it has the value $\frac{1}{1-r}$.

3. Flow chart for checking convergence of $\sum a_n$.
   i. Is $\lim a_n = 0$? If not, then it diverges.
   ii. If the terms aren’t eventually all positive or all negative, use the alternating test.
   iii. If all the terms are nonnegative, try the ratio test.
   iv. If ratio test fails, try the integral test if $a_n$ looks like an easily integrable function.
   v. If nothing works, try the comparison test and combining that with the other tests.

1.2 Integral test

1. If $f(x) \geq 0$ for $x \geq a$ and $f(x)$ is monotonically decreasing, then the expressions $\sum_{n=a}^{\infty} f(n)$ and $\int_{a}^{\infty} f(x) \, dx$ either both converge or both diverge.

2. From this, we see that $\sum_{n=1}^{\infty} n^{-p}$ diverges when $p = 1$, but converges when $p > 1$.

1.3 Comparison test

1. If $a_n, b_n \geq 0$, $|a_n| \leq b_n$, and $\sum b_n$ converges, then $\sum a_n$ converges as well.

2. If $a_n, b_n \geq 0$ and $\lim(a_n/b_n) = 1$, then the expressions $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

1.4 Ratio test

1. If $a_n \geq 0$ and $\lim(a_{n+1}/a_n) = L$, then the series $\sum a_n$ converges if $L < 1$ and diverges if $L > 1$. (If $L = 1$, then this test is inconclusive.)

1.5 Alternating test

1. The sum $\sum a_n$ converges conditionally if $\sum a_n$ converges, but $\sum |a_n|$ does not. (This can only happen if some of the $a_n$ are negative and some are positive.)

2. The alternating series test says that if $a_n \geq 0$ and $a_1 \geq a_2 \geq a_3 \geq \cdots$ with $\lim a_n = 0$, then the series $\sum (-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$ converges.

3. The sum $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is an example of conditional convergence: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$ even though $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$. 
2 Problems

1. Determine whether the following series converge or diverge.

(a) \( \sum_{n=1}^{\infty} \frac{2n}{n + 2} \)
(b) \( \sum_{n=0}^{\infty} \frac{n}{(n^2 + 1)^{3/2}} \)
(c) \( \sum_{n=0}^{\infty} \frac{n^2}{n^3 + 1} \)
(d) \( \sum_{n=0}^{\infty} ne^{-n} \)
(e) \( \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \)
(f) \( \sum_{n=0}^{\infty} \frac{\sin(n)}{2^n} \)
(g) \( \sum_{n=1}^{\infty} (-1)^n \cdot (e^{1/n} - 1) \)
(h) \( \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} \)
(i) \( \sum_{n=1}^{\infty} \left( 1 - \cos \left( \frac{1}{n} \right) \right) \)

3 Answers

1. (a) Diverges, (b) Converges, (c) Diverges, (d) Converges, (e) Converges, (f) Converges, (g) Converges, (h) Converges, (i) Converges