1 Lecture review

1.1 Vector fields

1. A scalar field associates a scalar to every point. This is the same thing as a function. For example if \( z = f(x, y) \), then this is an assignment of the value \( f(x, y) \) to the point \((x, y)\).

2. A scalar field of the form \( z = f(x, y) \) defines a surface. From this equation, one can obtain level curves. The level curve associated to the value \( c \) will be the curve of points \((x, y)\) such that \( f(x, y) = c \).

3. Given a scalar field of the form \( w = f(x, y, z) \), one can take level surfaces: the level surface associated to the value \( c \) will be the surface of points \((x, y, z)\) such that \( f(x, y, z) = c \).

4. A vector field associates a scalar to every point. Its components will be scalar fields.

5. Consider a vector field of the form \( \mathbf{F}(x, y) = f_1(x, y) \mathbf{i} + f_2(x, y) \mathbf{j} \). The vector field \( \mathbf{F} \) is built up from the two scalar fields \( f_1(x, y) \), \( f_2(x, y) \) (and the two vectors \( \mathbf{i}, \mathbf{j} \)).

1.2 Gradient

1. The gradient \( \nabla \) is an operator that turns scalar fields into vector fields.

<table>
<thead>
<tr>
<th>( \nabla f ) for two-variable functions ( z = f(x, y) )</th>
<th>( \nabla f ) for three-variable functions ( w = f(x, y, z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nabla f = \text{grad } f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} = f_x \mathbf{i} + f_y \mathbf{j} )</td>
<td>( \nabla f = \text{grad } f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k} )</td>
</tr>
<tr>
<td>perpendicular to level curves</td>
<td>normal to level surfaces</td>
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</table>

2. Warning: For a two variable function \( f \), notice that \( \nabla f \) has no \( \mathbf{k} \)-component. It only gives a vector on the contour map (as opposed to being a tangent direction on the surface).

3. Given a surface of the form \( f(x, y, z) = c \), this is the level surface of \( w = f(x, y, z) \) so a normal to the tangent plane of this surface is given by \( \nabla f \). This can often make computation of the tangent plane easier when given an implicit equation for the surface.

1.3 Directional derivatives

1. For a 2D or parametric curve, there is always an obvious “direction” in which differentiation always takes place: this is the positive \( x \)-direction for the 2D curve, and the positive \( t \)-direction for the parametric curve.

2. However, one needs to choose a direction when differentiating higher dimensional objects. The directional derivative is the notion of a derivative “in a direction.”

3. If \( f \) is a function and \( \mathbf{R} \) is a direction, then

\[
\nabla f \cdot \frac{\mathbf{R}}{||\mathbf{R}||}
\]

is the directional derivative of \( f \) in the direction \( \mathbf{R} \).
2 Problems

2.1 Tangent planes, linear approximation

1. (2B-1) Give the equation of the tangent plane to each of these surfaces at the point indicated.

(a) \( z = xy^2 \) at \((1, 1, 1)\). Use this tangent plane to approximate the value of \( z \) when \( x = 1.01, y = 1.01 \). How close is your approximation to the true value?

\[
\begin{align*}
\frac{\partial z}{\partial x} &= y^2 = 1, \\
\frac{\partial z}{\partial y} &= 2xy = 2, \\
\frac{\partial z}{\partial x} + 2(y-1) &= z-1 \\
z &= x + 2y - 2 \\
(1.01, 1.01) &\approx 1.01 + 2 \cdot 1.01 - 2 = 1.03 \\
\text{(actual value is } 1.03031) \\
\end{align*}
\]

(b) \( z = y^2/x \) at \((1, 2, 4)\). Use this tangent plane to approximate the value of \( z \) when \( x = 1.01, y = 2.01 \). How close is your approximation to the true value?

\[
\begin{align*}
\frac{\partial z}{\partial x} &= -\frac{y^2}{x^2} = -4, \\
\frac{\partial z}{\partial y} &= \frac{2y}{x} = 4, \\
-\frac{4(x-1)}{1.01} + 4 \left( \frac{y-2}{2.01} \right) &= z-4 \\
z &= -4x + 4y \\
(1.01, 2.01) &\approx -4 \cdot 1.01 + 4 \cdot 2.01 = 4 \\
\text{(actual value is } \approx 4.00001) \\
\end{align*}
\]

2. Consider the function \( z = \sqrt{x^2 + y^2} \).

(a) Describe the shape.

\[
\text{Cone}
\]

(b) Compute the equation of the tangent plane at an arbitrary point. This plane intersects the shape at more than just that point; what is the intersection?

\[
\frac{x-x_0}{\sqrt{x_0^2+y_0^2}} + \frac{y-y_0}{\sqrt{x_0^2+y_0^2}} = \frac{z-z_0}{\sqrt{x_0^2+y_0^2}}. \\
\text{Intersects in a whole line (line between } (x_0, y_0, z_0) \text{ and origin)}
\]

(c) What happens at the origin when you attempt to compute the tangent plane?

\[
\text{Have to divide by } 0. \\
\text{No well-defined tangent plane due to sharp cone point.}
\]

3. (2B-6) To determine the volume of a cylinder of radius around 2 and height around 3, about how accurately should the radius and height be measured for the error in the calculated volume not to exceed 0.1?

\[
V = \pi r^2 h \quad \frac{\partial V}{\partial r} = 2\pi r h \quad \frac{\partial V}{\partial h} = \pi r^2 = 4\pi \]

\[
\Delta V \approx 12\pi \Delta r + 4\pi \Delta h \leq 16\pi \varepsilon < 0.1 \
\text{if } \Delta r, \Delta h \leq \varepsilon < \frac{0.1}{16\pi} < 0.002
\]
2.2 Vector fields

1. (2A-1) Sketch five level curves for each of the following functions. Also, for a-d, sketch the portion of the graph of the function lying in the first octant; include in your sketch the traces of the graph in the three coordinate planes, if possible.

\[
\begin{align*}
1 - x - y & \quad \sqrt{x^2 + y^2} & x^2 + y^2 & \quad 1 - x^2 - y^2 & \quad x^2 - y^2 \\
\end{align*}
\]

2.3 Gradient, directional derivative

1. The gradient field always points in the direction of the greatest increase. How would you find the direction of the greatest decrease?

\[ -\nabla f \]

2. What is the name for the directional derivatives in the \( x \)-direction? The \( y \)-direction?

\[ \frac{df}{dx}, \frac{df}{dy} \]

3. Calculate the directional derivative of the temperature field \( T(x, y, z) = yx^2 + xz^2 + xyz \) at the point \((1, 1, 1)\) in each of the directions:

\[ \nabla T = \langle 2xy + z^2 + yz, x^2 + xz, 2xz + xy \rangle \]

(a) \((1, 0, 0)\)

\[ \langle 4, 2, 3 \rangle \cdot \langle 1, 0, 0 \rangle = 4 \]

(b) \((1, 1, 1)\)

\[ \langle 4, 2, 3 \rangle \cdot \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle = \frac{9}{\sqrt{3}} = 3\sqrt{3} \]

(c) \((0, 1, 1)\)

\[ \langle 4, 2, 3 \rangle \cdot \langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{5}{\sqrt{2}} \]
4. Compute and sketch the tangent planes to the ellipsoid \(x^2 + 2y^2 + 3z^2 = 6\) at the following points.

\[
\nabla f = 2x \hat{i} + 4y \hat{j} + 6z \hat{k}
\]

(a) \((0, 0, \sqrt{2})\)

\[
0(x-0) + 0(y-0) + 6\sqrt{2}(z-\sqrt{2}) = 0 \quad \quad z = \sqrt{2}
\]

(b) \((1, 1, 1)\)

\[
2(x-1) + 4(y-1) + 6(z-1) = 0 \quad \quad x+2y+3z = 6.
\]

5. (2D-2) For the following functions, each with a given point \(P\),

(i) \(w = \ln(4x - 3y)\), \((1, 1)\)

(ii) \(w = xy + yz + zx\), \((1, -1, 2)\)

(iii) \(w = \sin^2(t-u)\), \((\pi/4, 0)\)

(a) compute the gradient of the function at \(P\)

(b) find the directions where the magnitude of the directional derivative is maximized

(c) find the directions where the magnitude of the directional derivative is zero

\[
\begin{align*}
\text{(a)} & \quad \nabla f = \left\langle \frac{4}{4x-3y}, \frac{-3}{4x-3y} \right\rangle \\
\nabla f|_P = \left\langle \frac{1}{1}, \frac{3}{1} \right\rangle \\
\text{(b)} & \quad \nabla f = \left\langle y+z, 2+x, x+y \right\rangle \\
\nabla f|_P = \left\langle 1, 3, 0 \right\rangle \\
\text{(c)} & \quad \nabla f = \left\langle \sin(2(t-u)), \sin(2(t-u)) \right\rangle \\
\nabla f|_P = \left\langle \frac{1}{1}, \frac{1}{1} \right\rangle
\end{align*}
\]

6. Suppose that a directional derivative of \(z = f(x, y)\) at a point \((x_0, y_0, z_0)\) in a direction \(\mathbf{u}\) is zero. Explain why \(\mathbf{u}\) must be a tangent vector to the level curve given by \(f(x, y) = z_0\). Check that this is true for the first and third examples in the previous problem. (It does not make sense for the second; why? Challenge: How can this question be modified to make sense for the second example of the previous problem?)

\[
\nabla f \cdot \mathbf{R}/|\mathbf{R}| = 0 \rightarrow \mathbf{R} \text{ is } \perp \text{ to } \nabla f. \quad \text{But } \nabla f \text{ is normal to level curve } \rightarrow \mathbf{R} \text{ is tangent to level curve}
\]

(i) \(\ln(4x-3y) = 0\) tangent line at \((1,1)\) is \(y = \frac{4}{3}x - \frac{1}{3}\).

(ii) \(\sin^2(t-u) = \frac{1}{2} \rightarrow t-u = \frac{\pi}{4}\) tangent line at \((1,1)\) is \(u = t - \pi/4\).

7. (2D-9) [Done on the board during recitation.]