1 Lecture review

1.1 Calculating partial derivatives

If \( w \) is a function of several variables, say \( w = f(x, y, z) \), then \( \frac{\partial f}{\partial x} \) is the partial derivative of \( f \) with respect to \( x \); to evaluate this, treat \( y \) and \( z \) as constants and differentiate with respect to \( x \).

1.2 Geometric meaning

1. Tangent lines

(a) Single variable: if \( y = y(x) \) then \( \frac{dy}{dx} \bigg|_{x=x_0} \) is the slope of the tangent line at \( x = x_0 \).

(b) Multivariable: if \( z = z(x, y) \) then \( \frac{dz}{dx} \bigg|_{(x,y)=(x_0,y_0)} \) is the slope of the tangent line in the \( x \) direction at \( (x,y) = (x_0,y_0) \).

(c) Multivariable: if \( z = z(x, y) \) then \( \frac{dz}{dy} \bigg|_{(x,y)=(x_0,y_0)} \) is the slope of the tangent line in the \( y \) direction at \( (x,y) = (x_0,y_0) \).

2. Tangent planes are a multivariable concept. Since \( \mathbf{T}_x = \hat{i} + \frac{\partial z}{\partial x} \hat{k} \) and \( \mathbf{T}_y = \hat{j} + \frac{\partial z}{\partial y} \hat{k} \) are both tangent vectors, the vector

\[
\mathbf{T}_y \times \mathbf{T}_x = \frac{\partial z}{\partial x} \hat{i} + \frac{\partial z}{\partial y} \hat{j} - \hat{k}
\]

is a normal vector to the tangent plane.

(a) If \( z = f(x, y) \), then an equation for the tangent plane at \( (x_0, y_0, z_0) \) is

\[
z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

(b) This equation can be used to approximate the function \( f \) near the point \( (x, y) = (x_0, y_0) \):

\[
f(x, y) \approx z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

2 Problems

2.1 Partial differentiation

1. Consider the ideal gas law \( PV = kT \). Compute the following.

\[
\frac{\partial P}{\partial T} \quad \frac{\partial T}{\partial P} \quad \frac{\partial P}{\partial V} \quad \frac{\partial V}{\partial P}
\]

2. Let \( f(x, y) = x^2y^3 + \sin(2x) \). Compute

\[
f_x \quad f_y \quad f_{xx} \quad f_{yy} \quad f_{xy} \quad f_{yx}
\]

What do you notice about the last two of these expressions?

3. Let \( f(x, y, z) = ze^{x+y} + x^2yz \). Compute \( f_x, f_{xy}, f_{xyz} \).

4. (Challenge) Let

\[
f(x, y) = \begin{cases} 
\frac{xy(x^2-y^2)}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\
0 & \text{for } (x, y) = (0, 0).
\end{cases}
\]

Decide whether the values \( f, f_x, f_y, f_{xy}, f_{yx} \) exist at \( (0, 0) \); if they do, compute them. What do you notice about \( f_{xy} \) and \( f_{yx} \) at \( (0, 0) \)?
2.2 Tangent planes

1. Consider the function \( z = \sqrt{9 - x^2 - y^2} \).
   (a) Describe this shape.
   (b) Using geometric intuition, what would be a unit normal vector at the point (2, 1, 2)?
   (c) Using partial differentiation, verify that your answer in the previous part is correct. Compute the tangent plane at the point (2, 1, 2).

2. Consider the function \( z = 3x + 4y \). Compute the tangent plane at the point (1, 1). What do you notice?

3. Consider the function \( z = \sqrt{x^2 + y^2} \).
   (a) Describe the shape.
   (b) Compute the equation of the tangent plane at an arbitrary point. This plane intersects the shape at more than just that point; what is the intersection?
   (c) What happens at the origin when you attempt to compute the tangent plane?

4. Suppose the tangent plane to a surface given by \( z = f(x, y) \) at the point \( P \) given by \( x = 1, y = 2 \) is \( x + 2y + 3z = 4 \).
   (a) What is \( f(1, 2) \)?
   (b) What is \( f_x(1, 2) \)?
   (c) What is \( f_y(1, 2) \)?
   (d) Write all possible tangent vectors to this surface at \( P \). (Hint: find the normal vector to the surface at \( P \))

2.3 Linear approximation

1. (2B-1) Give the equation of the tangent plane to each of these surfaces at the point indicated.
   (a) \( z = xy^2 \) at (1, 1, 1). Use this tangent plane to approximate the value of \( z \) when \( x = 1.01, y = 1.01 \). How close is your approximation to the true value?
   (b) \( z = y^2/x \) at (1, 2, 4). Use this tangent plane to approximate the value of \( z \) when \( x = 1.01, y = 2.01 \). How close is your approximation to the true value?

2. (2B-3) Using the approximation formula, find the approximate change in the hypotenuse of a right triangle, if the legs, initially of length 3 and 4, are each increased by .010. Compute the actual increase and compare it to the approximation.

3. (2B-6) To determine the volume of a cylinder of radius around 2 and height around 3, about how accurately should the radius and height be measured for the error in the calculated volume not to exceed 0.1?