1 Lecture review

1.1 Calculating partial derivatives

If \( w \) is a function of several variables, say \( w = f(x, y, z) \), then \( \frac{\partial f}{\partial x} \) is the partial derivative of \( f \) with respect to \( x \); to evaluate this, treat \( y \) and \( z \) as constants and differentiate with respect to \( x \).

1.2 Geometric meaning

1. Tangent lines
   (a) Single variable: if \( y = y(x) \) then \( \frac{dy}{dx} \) is the slope of the tangent line at \( x = x_0 \).
   (b) Multivariable: if \( z = z(x, y) \) then \( \frac{dz}{dx} \) is the slope of the tangent line in the \( x \) direction at \( (x, y) = (x_0, y_0) \).
   (c) Multivariable: if \( z = z(x, y) \) then \( \frac{dz}{dy} \) is the slope of the tangent line in the \( y \) direction at \( (x, y) = (x_0, y_0) \).

2. Tangent planes are a multivariable concept. Since \( \mathbf{T}_x = \hat{i} + \frac{\partial z}{\partial x} \hat{k} \) and \( \mathbf{T}_y = \hat{j} + \frac{\partial z}{\partial y} \hat{k} \) are both tangent vectors, the vector
   \[ \mathbf{T}_y \times \mathbf{T}_x = \frac{\partial z}{\partial x} \hat{i} + \frac{\partial z}{\partial y} \hat{j} - \hat{k} \]
   is a normal vector to the tangent plane.
   (a) If \( z = f(x, y) \), then an equation for the tangent plane at \( (x_0, y_0, z_0) \) is
   \[ z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \]
   (b) This equation can be used to approximate the function \( f \) near the point \( (x, y) = (x_0, y_0) \):
   \[ f(x, y) \approx z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \]

2 Problems

2.1 Partial differentiation

1. Consider the ideal gas law \( PV = kT \). Compute the following.

   \[
   \frac{\partial P}{\partial T} = \frac{k}{V}, \quad \frac{\partial T}{\partial P} = \frac{V}{k}, \quad \frac{\partial P}{\partial V} = -\frac{kT}{V^2}, \quad \frac{\partial V}{\partial P} = -\frac{kT}{p^2}
   \]

2. Let \( f(x, y) = x^2y^3 + \sin(2x) \). Compute

   \[
   f_x, \quad f_y, \quad f_{xx}, \quad f_{yy}, \quad f_{xy} = f_{yx} = 6xy^2
   \]

   What do you notice about the last two of these expressions?
3. Let \( f(x, y, z) = ze^{x+y} + x^2yz \). Compute \( f_x, f_{xy}, f_{xyz} \).

\[
\begin{align*}
f_x &= ze^{x+y} + 2yzx \\
f_{xy} &= ze^{x+y} + 2xz \\
f_{xyz} &= e^{x+y} + 2x 
\end{align*}
\]

4. (Challenge) Let

\[
f(x, y) = \begin{cases} 
\frac{y(x^2-y^2)}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\
0 & \text{for } (x, y) = (0, 0).
\end{cases}
\]

Decide whether the values \( f, f_x, f_y, f_{xy}, f_{yx} \) exist at \((0, 0)\); if they do, compute them. What do you notice about \( f_{xy} \) and \( f_{yx} \) at \((0, 0)\)?

2.2 Tangent planes

1. Consider the function \( z = \sqrt{9-x^2-y^2} \).
   
   (a) Describe this shape.
   
   Hemispher of radius 3, \( z \geq 0 \)
   
   (b) Using geometric intuition, what would be a unit normal vector at the point \((2, 1, 2)\)?
   
   \( \pm \left( \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right) \) (radial direction)
   
   (c) Using partial differentiation, verify that your answer in the previous part is correct. Compute the tangent plane at the point \((2, 1, 2)\).

   \[
   \begin{align*}
   f_x &= 3 \\
   f_y &= 4 \\
   f_z &= \frac{3(x-1) + 4(y-1) - z - 7}{2} \\
   \text{tangent plane} &= \text{surface}
   \end{align*}
   \]

2. Consider the function \( z = 3x + 4y \). Compute the tangent plane at the point \((1, 1)\). What do you notice?

3. Consider the function \( z = \sqrt{x^2 + y^2} \).
   
   (a) Describe the shape.
   
   Cone
   
   (b) Compute the equation of the tangent plane at an arbitrary point. This plane intersects the shape at more than just that point; what is the intersection?

   \[
   \begin{align*}
   f_x &= \frac{x}{\sqrt{x^2+y^2}} \\
   f_y &= \frac{y}{\sqrt{x^2+y^2}} \\
   \frac{x_0}{\sqrt{x_0^2+y_0^2}} (x-x_0) + \frac{y_0}{\sqrt{x_0^2+y_0^2}} (y-y_0) &= z - z_0 \\
   \text{Intersects in a whole line (line between } (x_0, y_0, z_0) \text{ and origin)}
   \end{align*}
   \]
(c) What happens at the origin when you attempt to compute the tangent plane?

- Have to divide by 0.
- No well-defined tangent plane due to sharp cone point.

4. Suppose the tangent plane to a surface given by \( z = f(x, y) \) at the point \( P \) given by \( x = 1, y = 2 \) is \( x + 2y + 3z = 4 \).

(a) What is \( f(1, 2) \)?

\[
\begin{align*}
\hat{f}(1, 2) &= \frac{4}{3} - \frac{1}{3} x - \frac{2}{3} y \\
&= \frac{2}{3} - \frac{2}{3} (2) = -\frac{1}{3}
\end{align*}
\]

(b) What is \( f_x(1, 2) \)?

\[
\hat{f}_x (1, 2) = -\frac{1}{3}
\]

(c) What is \( f_y(1, 2) \)?

\[
\hat{f}_y (1, 2) = -\frac{2}{3}
\]

(d) Write all possible tangent vectors to this surface at \( P \). (Hint: find the normal vector to the surface at \( P \))

\[
\text{normal} = \langle 1, 2, 3 \rangle \\
\text{tangent} = \langle a, b, c \rangle \\
\angle (1, 2, 3) \cdot (a, b, c) = 0 \\
a + 2b + 3c = 0
\]

2.3 Linear approximation

1. (2B-1) Give the equation of the tangent plane to each of these surfaces at the point indicated.

(a) \( z = xy^2 \) at \((1, 1, 1)\). Use this tangent plane to approximate the value of \( z \) when \( x = 1.01, y = 1.01 \). How close is your approximation to the true value?

\[
\begin{align*}
f_x &= y^2 \\
f_y &= 2xy \\
(1.01 - 1) + 2(1.01 - 1) &= z - 1 \\
z &= x + 2y - 2 \\
f(1.01, 1.01) &= 1.01 + 2(1.01) - 2 = 1.03 (\text{actual value}) \\
f(1.01, 1.01) &\approx 1.03 (1.03031 \text{ is close})
\end{align*}
\]

(b) \( z = \frac{y^2}{x} \) at \((1, 2, 4)\). Use this tangent plane to approximate the value of \( z \) when \( x = 1.01, y = 2.01 \). How close is your approximation to the true value?

\[
\begin{align*}
f_x &= -y^2/x^2 \\
f_y &= 2y/x \\
-4(1.01 - 1) + 4(2.01 - 2) &= z - 4 \\
z &= -4x + 4y \\
f(1.01, 2.01) &\approx -4(1.01) + 4(2.01) = 4 \text{ (actual value)} \\
f(1.01, 2.01) &\approx 4 \text{ (4.0001 is close)}
\end{align*}
\]

2. (2B-3) Using the approximation formula, find the approximate change in the hypotenuse of a right triangle, if the legs, initially of length 3 and 4, are each increased by \( .010 \). Compute the actual increase and compare it to the approximation.

\[
\begin{align*}
f &\approx \sqrt{x^2 + y^2} \\
\frac{\partial f}{\partial x} &= \frac{x}{\sqrt{x^2 + y^2}} = \frac{3}{5} \\
\frac{\partial f}{\partial y} &= \frac{y}{\sqrt{x^2 + y^2}} = \frac{4}{5} (\text{actual change}) \\
f &\approx 5 = \frac{3}{5}(x-3) + \frac{4}{5}(y-4) \\
5 &\approx \frac{3}{5}x + \frac{4}{5}y \\
f &\approx \frac{3}{5}0.01 + \frac{4}{5}0.01 = 0.014 \text{ (is close)}
\end{align*}
\]

3. (2B-6) To determine the volume of a cylinder of radius around 2 and height around 3, about how accurately should the radius and height be measured for the error in the calculated volume not to exceed 0.1?

\[
\begin{align*}
V &= \pi r^2 h \\
\frac{\partial V}{\partial r} &= 2\pi rh \\
\frac{\partial V}{\partial h} &= \pi r^2 \\
\Delta V &= 12\pi \Delta r + 4\pi \Delta h \\
\Delta V &\leq 16\pi \varepsilon \leq 0.1 \\
\varepsilon &\leq \frac{0.1}{16\pi} < 0.002
\end{align*}
\]