1 Lecture review

We will work with a parametric curve given by
\[ \mathbf{r}(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k} \]

1. The velocity vector is
\[ \mathbf{V}(t) = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} + \frac{dz}{dt} \hat{k}. \]

The velocity vector is always tangent to the curve.

2. The speed is the magnitude of the velocity vector. That is,
\[ \frac{ds}{dt} = |\mathbf{V}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \]

3. The acceleration vector is
\[ \mathbf{a}(t) = \frac{d^2x}{dt^2} \hat{i} + \frac{d^2y}{dt^2} \hat{j} + \frac{d^2z}{dt^2} \hat{k}. \]

4. The unit tangent vector is the normalized velocity vector. It is given by
\[ \mathbf{T}(t) = \frac{\mathbf{V}(t)}{|\mathbf{V}(t)|} = \frac{\frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} + \frac{dz}{dt} \hat{k}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}. \]

Therefore, the unit vector is always tangent to the curve and it has unit length.

5. The unit normal vector to a 2D parametric curve is defined to be
\[ \mathbf{N}(t) = \frac{\frac{dy}{dt} \hat{i} + \frac{dx}{dt} \hat{j}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}. \]

It is perpendicular to the unit tangent vector and it has unit length.

6. We have the following relationship between three of the aforementioned quantities.
\[ \mathbf{V}(t) = \frac{ds}{dt} \cdot \mathbf{T}(t) \]

7. The reason we denote the speed by \( ds/dt \) is because if one integrates it as follows
\[ S(t) = \int_0^t \frac{ds}{dt} \, dt = \int_0^t |\mathbf{V}(t)| \, dt, \]

then this is the arc length function; it is the length of the curve from time 0 to time \( t \). To obtain the arc length from \( t = t_1 \) to \( t = t_2 \), use
\[ S(t_2) - S(t_1) = \int_{t_1}^{t_2} \frac{ds}{dt} \, dt = \int_{t_1}^{t_2} |\mathbf{V}(t)| \, dt \]
2 Problems

1. Consider the curve in the $xy$-plane defined by the equation $y^2 = x^3$

(a) Write down two different parametrizations for the part of the curve between $(0, 0)$ and $(1, 1)$.

First \[ \langle x, y \rangle = \langle t^2, t^3 \rangle \]

Second \[ \langle x, y \rangle = \langle t, t^{3/2} \rangle \]

(b) For each parametrization of the curve you wrote in the previous part, compute

\[
\begin{align*}
\vec{V}(t) &= \langle 2t, 3t^2 \rangle \\
\vec{a}(t) &= \langle 2, 6t \rangle \\
\frac{ds}{dt} &= \sqrt{4t^2 + 9t^4} \\
\vec{T}(t) &= \langle 2, 3t \rangle \\
\vec{N}(t) &= \langle -3t, 2 \rangle
\end{align*}
\]

(c) Compute the arc length of this curve from $(0, 0)$ to $(1, 1)$ in two ways by using the two parametrizations you created in the first part. Do your answers agree?

First \[ \int_0^1 \sqrt{4t^2 + 9t^4} \, dt = \int_0^1 \sqrt{4t^2 + 9t^4} \, dt = \int_0^1 \sqrt{1 + \frac{9}{4}t^2} \, du = \frac{\sqrt{13}}{4} \int_0^1 \frac{du}{\sqrt{1 + \frac{9}{4}u^2}} \]

Second \[ \int_0^1 \sqrt{1 + \frac{9}{4}t} \, dt \]

2. Consider the curve defined by the parametric equation $\vec{r}(t) = t \hat{i} + \frac{1}{2} t^2 \hat{j}$.

(a) Sketch the curve.

(b) Find the velocity and acceleration along the curve.

\[ \vec{V}(t) = \langle 1, t \rangle \quad \vec{a}(t) = \langle 0, 1 \rangle \]

(c) Find the unit tangent and normal vectors to the curve. Check that they are orthogonal.

\[ \vec{T}(t) = \langle 1, t \rangle \quad \vec{N}(t) = \langle -t, 1 \rangle \quad \vec{V}(t) \cdot \vec{N}(t) = \frac{(1)(-t) + (1)(1)}{\sqrt{1 + t^2}} = 0 \]

(d) Find the arc length traced out between $t = 0$ and $t = 10$.

\[ L = \int_0^{10} \sqrt{1 + t^2} \, dt = \frac{\sqrt{1 + t^2}}{t} \bigg|_0^{10} + \frac{1}{2} t \bigg|_0^{10} = 10\sqrt{101} - \ln \left| \frac{10\sqrt{101} - \frac{1}{2} \ln 10}{\frac{1}{2} \ln 10} \right| \]

\[ = 10\sqrt{101} + \ln \left| \frac{10\sqrt{101} + \ln 10}{\ln 10} \right| \]

\[ = 5 \sqrt{101} + \frac{1}{2} \ln \left| 10 \cdot 10\sqrt{101} \right| \]
3. Consider circular motion given by the parametric equation

\[ r(t) = a \cos \omega t \, \hat{i} + a \sin \omega t \, \hat{j}, \]

where \( t \) is time. Calculate velocity, acceleration, speed, curvature, radius of curvature and the tangent and normal vectors. Calculate the arc length traced out in one revolution (from \( t = 0 \) to \( t = 2\pi \)).

\[ \vec{v}(t) = \left<-a \omega \sin \omega t, a \omega \cos \omega t \right> \]
\[ \vec{a}(t) = \left<-a \omega^2 \cos \omega t, -a \omega^2 \sin \omega t \right> \]
\[ \kappa(t) = \frac{1}{|\vec{v} \times \vec{a}|/|\vec{v}|^3} = \frac{a \omega^3}{a^2 \omega^3} = \frac{1}{a} \]
\[ R(t) = \frac{1}{\kappa(t)} = \frac{1}{(1/a)} = a \]
\[ L = \int_0^{2\pi} \sqrt{a^2 \omega^2 (\sin^2 \omega t + \cos^2 \omega t)} \, dt \]
\[ = 2\pi a \omega. \]

4. (1J-2) Let

\[ \vec{r}(t) = \frac{1}{1+t^2} \hat{i} + \frac{t}{1+t^2} \hat{j} \]

be the position vector for a motion.

(a) Calculate \( \vec{V}(t), |ds/dt|, \) and \( \vec{T}(t). \)

\[ |ds/dt| = \sqrt{(\frac{t^2}{(1+t^2)^2}) + (\frac{1}{(1+t^2)^2})^2} = \frac{1}{(1+t^2)} \]
\[ \vec{T}(t) = \left< \frac{-2t}{(1+t^2)^2}, \frac{1-t^2}{(1+t^2)^2} \right> \]

(b) At what point in the speed greatest? smallest? speed = \( 1/(1+t^2) \)

maximized when \( t = 0 \), \( \text{max speed} \) at \( t = 0 \)

minimum speed \( \text{never reaches} \) \( \text{speed} \to 0 \) as \( t \to \infty \)

(c) Find the \( xy \)-equation of this parametric curve, and describe it geometrically.

\[ x = \frac{1}{1+t^2}, \quad y = \frac{t}{1+t^2}. \]
\[ \frac{y}{x} = t \to x = \frac{1}{1+t^2} = \frac{1}{1(0^2)} = \frac{0^2}{x^2+y^2} \Rightarrow x^2+y^2 = \text{CIRCLE}. \]

5. (1J-9) A point \( P \) is moving in space, with position vector

\[ \vec{r}(t) = \overrightarrow{OP} = 3 \cos t \, \hat{i} + 5 \sin t \, \hat{j} + 4 \cos t \, \hat{k}. \]

(a) Show it moves on the surface of a sphere.

\[ |\vec{r}(t)| = 5, \text{ so always on surface of sphere of radius } 5 \]

(b) Show its speed is constant.

\[ |\vec{V}(t)| = \sqrt{(-3 \sin t)^2 + (5 \cos t)^2 + (-4 \sin t)^2} = 5. \]

(c) Show the acceleration is directed toward the origin.

\[ \vec{a}(t) = \left<-3 \cos t, -5 \sin t, -4 \cos t \right> \quad \text{points toward origin.} \]
\[ \text{points toward origin.} \]
\[ \text{points toward origin.} \]

(d) Show it moves in a plane through the origin.

\[ \text{always lies in plane } 4x - 3z = 0. \]

(e) Describe the path of the point.

\[ \text{given by an ellipse in the above plane.} \]