Recitation 3: Matrix Arithmetic
18.02 Section R21
September 13, 2017

1 Lecture review

1.1 Notation
1. When one says “A is an $m \times n$ matrix,” that means it has $m$ rows and $n$ columns.

1.2 Significance of determinant
1. Geometrically, the determinant computes the signed volume of the parallelepiped given by the columns of the matrix.
   
   (a) In particular, the $2 \times 2$ matrix is computing the area of a parallelogram.
   
   (b) The $3 \times 3$ matrix computes the volume of a parallelepiped (the three-dimensional analogue of the parallelogram.)

1.3 Matrix addition
1. It only makes sense to add matrices of the same dimension. If $A$ and $B$ are both $m \times n$ matrices, then $A + B$ will be an $m \times n$ matrix.

2. Matrix addition is performed entrywise.

1.4 Matrix multiplication
1. If $A$ is an $m \times n$ matrix and $B$ is an $n \times k$ matrix, then $AB$ is an $m \times k$ matrix. In order for $AB$ to make sense, the number of columns of $A$ must equal the number of rows of $B$.

2. Think, “pouring rows down columns.”

3. Example problems.

   (a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$. Then $AB = \begin{pmatrix} 5 + 12 \\ 15 + 24 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}$.

   (b) $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$. Then $AB = \begin{pmatrix} 1 & 3 \\ 1 & 6 \\ 2 & 5 \end{pmatrix}$.

1.5 Matrix inversion
1. This means “inverting (undoing) the operation of matrix multiplication.”

2. It only makes sense to invert square matrices; that is, those which have the same number of rows as columns. If $A$ is an $n \times n$ matrix, then $A^{-1}$ is also an $n \times n$ matrix.

3. Method for computing (in this particular order!)
   
   (a) Minors (determinants of submatrices)
   
   (b) Cofactors (signs)
   
   (c) Adjoint (transpose)
   
   (d) Divide by determinant

4. Example: $A = \begin{pmatrix} -5 & -1 & -1 \\ 4 & 10 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. Then $A^{-1} = \begin{pmatrix} 1 & -2 & 7 \\ 2 & -3 & 11 \\ -8 & 13 & -46 \end{pmatrix}$. Check that $AA^{-1}$ and $A^{-1}A$ are the identity matrix.
2 Problems

1. Define

\[
A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}
\]

(a) For each of the expressions listed below, determine (a) whether it is a valid matrix expression and (b) if it is, the dimensions of the result.

\[
\begin{align*}
A^{-3} & \quad ABCDE & \quad CAB & \quad CE^{-1}D & \quad AB + CD \\
BAD & \quad ABCD & \quad C^{-1}D^{-1} & \quad DC - E^2 & \quad DC + BA \\
(CD)^{-1} & \quad D^2 & \quad D^{-1}C^{-1} & \quad BD - CE & \quad BAB^{-1}
\end{align*}
\]

(b) Compute the following.

\[
CD \quad DC \quad E^{-1} \quad B^{-1} + A \quad AB - BA
\]

2. Let

\[
F = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}
\]

(a) Compute \(FG\) and \((FG)^{-1}\).

(b) Compute \(F^{-1}\) and \(G^{-1}\).

(c) Compute \(F^{-1}G^{-1}\). Does it agree with your answer for \((FG)^{-1}\) from the first part?

(d) Compute \(G^{-1}F^{-1}\). Does it agree with your answer for \((FG)^{-1}\) from the first part?

3. Let

\[
X = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}, \quad Y = \begin{pmatrix} y_{11} \\ y_{21} \end{pmatrix}
\]

(a) Compute \(XY\). How is this related to dot product?

(b) Compute \(YX\). Does \(XY = YX\)?

(c) Does \(\det(XY) = \det(YX)\)?

(d) Do \(\det(X)\) and \(\det(Y)\) make sense? Explain.

4. Let

\[
X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}
\]

(a) Compute \(XY\).

(b) Compute \(YX\). Does \(XY = YX\)?

(c) Does \(\det(XY) = \det(YX)\)?

(d) Compute \(\det(X) \cdot \det(Y)\). Does this match your answer from the previous part?

(e) Recall that \(I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\). Does \(\det(I + XY) = \det(I + YX)\)?

(f) (Challenge) Suppose \(Z\) is a \(2 \times 2\) matrix such that \(XZ = ZX\). Prove that \(\det(Z + XY) = \det(Z + YX)\).

5. (1F-3). Find all \(2 \times 2\) matrices \(W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) such that

\[
W^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]