1 Lecture review

1.1 Surface integrals, flux

1. Recall that for a surface \( z = f(x, y) \) we have

\[
\iint_{S} \mathbf{F} \cdot d\mathbf{S} = \iint_{R} \mathbf{F} \cdot (-f_x \hat{i} - f_y \hat{j} + \hat{k}) \, dA
\]

2. Define \( g(x, y, z) = z - f(x, y) = 0 \). Then

\[\nabla g = -f_x \hat{i} - f_y \hat{j} + \hat{k} \text{ is normal to } S\]

\[\hat{n} = \frac{\nabla g}{|\nabla g|} \text{ is the unit normal.}\]

\[dS = |\nabla g| \, dA = \sqrt{f_x^2 + f_y^2 + 1} \, dA\]

\[d\mathbf{S} = \hat{n} \, dS = \nabla g \, dA = (-f_x \hat{i} - f_y \hat{j} + \hat{k}) \, dA\]

1.2 Divergence theorem

1. Recall the 2D version: for a closed, simple, piecewise smooth, positively oriented curve \( C \) and a vector field \( \mathbf{F} \) we have

\[
\oint_{C} \mathbf{F} \cdot \hat{n} \, ds = \iint_{R} \nabla \cdot \mathbf{F} \, dA
\]

(We also called this the normal form of Green’s theorem.)

2. The 3D version is as follows. Let \( S \) be a closed piecewise smooth surface bounding a space region \( D \) with outward unit normal \( \hat{n} \). Then for a vector field \( \mathbf{F} \) we have

\[
\iint_{S} \mathbf{F} \cdot \hat{n} \, dS = \iiint_{D} \nabla \cdot \mathbf{F} \, dV
\]
2 Problems

1. Compute the flux of the vector field $\mathbf{F} = -x\mathbf{i} - y\mathbf{j} + 3z\mathbf{k}$ across the portion of the unit sphere lying in the first octant (that is, $x, y, z \geq 0$). Check that your answer is correct by applying the divergence theorem.

2. Verify that the divergence theorem holds for $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j} + z\mathbf{k}$ and $D$ the solid cylinder $y^2 + z^2 \leq 9, 0 \leq x \leq 2$.

3. Use the divergence theorem to calculate the flux of $\mathbf{F} = (2x^3 + y^3)\mathbf{i} + (y^3 + z^3)\mathbf{j} + 3y^2 z\mathbf{k}$ through $S$, the surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the $xy$-plane.

4. Verify that the divergence theorem holds for $\mathbf{F} = y^2 z^3 \mathbf{i} + 2yz\mathbf{j} + 4z^2 \mathbf{k}$ and $D$ is the solid enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$.

5. Use the divergence theorem to calculate the flux of $\mathbf{F} = z^2 x\mathbf{i} + \left(\frac{1}{3} y^3 + \tan z\right)\mathbf{j} + (x^2 z + y^2)\mathbf{k}$ through the top half of the unit sphere centered at the origin.

6. Find the closed surface $S$ in the region $z \geq 0$ through which the flux of $\mathbf{F} = (y^3 - xy^2 - x^3)\mathbf{i} + (6y + \sin z)\mathbf{j} - (z^2 + \log(x^2 + 1))\mathbf{k}$ is maximal.

7. Use the divergence theorem to calculate

$$\iint_S (2x + 2y + z^2) dS$$

where $S$ is the sphere $x^2 + y^2 + z^2 = 1$. 