1 Lecture review

1.1 Dot product
1. \( \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| \cdot |\mathbf{B}| \cdot \cos \theta \), where \( \theta \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \).
2. In coordinates, \((a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3\).
3. Dot product takes as input two vectors and outputs a scalar.

1.2 Cross product
1. \( \mathbf{A} \times \mathbf{B} = (|\mathbf{A}| \cdot |\mathbf{B}| \cdot \sin \theta) \cdot \mathbf{n} \), where \( \theta \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \) and \( \mathbf{n} \) is a unit normal to \( \mathbf{A} \) and \( \mathbf{B} \) whose direction is determined by the right hand rule.
2. In coordinates, \((a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)\). The following may be easier to remember:
   \[
   (a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) \times (b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}) = \begin{vmatrix}
   \mathbf{i} & \mathbf{j} & \mathbf{k} \\
   a_1 & a_2 & a_3 \\
   b_1 & b_2 & b_3 
   \end{vmatrix}
   \]
3. Cross product takes as input two vectors and outputs another vector.

1.3 Determinants
1. Here are formulas for \(2 \times 2\) and \(3 \times 3\) determinants.
   \[
   \begin{vmatrix}
   a & b \\
   c & d 
   \end{vmatrix} = ad - bc
   \]
   \[
   \begin{vmatrix}
   a_1 & a_2 & a_3 \\
   b_1 & b_2 & b_3 \\
   c_1 & c_2 & c_3 
   \end{vmatrix} = (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (a_3 b_2 c_1 + a_2 b_1 c_3 + a_1 b_3 c_2).
   \]
2. Handy facts for calculating determinants
   (a) Exchanging two rows (resp. column) flips sign of determinant.
   (b) Determinant is zero if an entire row (resp. column) is zero.
   (c) Adding a multiple of one row (resp. column) to another row (resp. column) does not affect determinant.
3. In general, use cofactor expansion for determinants of higher dimensional matrices after transforming it using “handy facts for calculating determinants” to put it into a simpler form.
4. Geometrically, the determinant computes the signed volume of the parallelepiped given by the columns of the matrix.
   (a) In particular, the \(2 \times 2\) matrix is computing the area of a parallelogram.
   (b) The \(3 \times 3\) matrix computes the volume of a parallelepiped (the three-dimensional analogue of the parallelogram.)
2 Problems

2.1 Conceptual

1. What is
   (a) $\vec{A} \cdot \vec{A}$?
   (b) $\vec{A} \times \vec{A}$?

2. Suppose you are given $\vec{A} \cdot \vec{B}$ and $\vec{A} \times \vec{B}$ (but not $\vec{A}$, $\vec{B}$ themselves).
   (a) How do you determine if $\vec{A}$ and $\vec{B}$ are perpendicular?
   (b) How do you determine if $\vec{A}$ and $\vec{B}$ are parallel?
   (c) How do you determine $\theta$, the angle between $\vec{A}$ and $\vec{B}$?
   (d) Give an example to demonstrate why this information is not sufficient to determine $\vec{A}$ and $\vec{B}$.

3. Let $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$.
   (a) Compute $\vec{v} \times \vec{w}$.
   (b) Compute the determinant of the matrix whose first column is $\vec{v}$ and whose second column is $\vec{w}$.
   (c) Explain how these two answers are related. What is the geometric quantity both are computing?

2.2 Computational

1. Compute the following.
   (a) $(1, 2) \cdot (3, 4)$
   (b) $(1, 2, 3) \cdot (4, 5, 6)$
   (c) $(2, 1) \times (1, 4)$
   (d) $(1, 2, 3) \times (4, 5, 6)$
   (e) $\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}$. How is this related to part (c)? (Hint: 2.1.3(c).)

2. Use cofactor expansion to compute
   $\begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & 5 \end{vmatrix}$, $\begin{vmatrix} 0 & 0 & 1 \\ -2109 & 3 & 4321 \\ 4 & 0 & 3210 \end{vmatrix}$

3. Use any method to compute the following determinants.
   $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$, $\begin{vmatrix} 7 & 3 & -9 \\ 1 & 2 & 0 \\ -5 & 6 & 8 \end{vmatrix}$, $\begin{vmatrix} 1 & 131 & 122 & 113 \\ 0 & 2 & 232 & 223 \\ 0 & 0 & 3 & 333 \\ 0 & 0 & 0 & 4 \end{vmatrix}$, $\begin{vmatrix} 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{vmatrix}$

4. (Challenge) Compute
   $\begin{vmatrix} x_1 \\ x_1 \end{vmatrix}$, $\begin{vmatrix} x_1 & x_1 \\ x_1 & x_2 \end{vmatrix}$, $\begin{vmatrix} x_1 & x_1 & x_1 \\ x_1 & x_2 & x_2 \\ x_1 & x_2 & x_3 \end{vmatrix}$, $\begin{vmatrix} x_1 & x_1 & x_1 & x_1 \\ x_1 & x_2 & x_2 & x_2 \\ x_1 & x_2 & x_3 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{vmatrix}$

What is the pattern? Can you generalize it?