1 Lecture review

1. Define the operator
   \[ \nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j}. \]

2. Gradient = \( \nabla f \). It turns scalar fields/functions \( f \) into vector fields \( \nabla f \). Its direction is that of the greatest increase of \( f \). Its magnitude is how quickly \( f \) is increasing in that direction.
   
   In coordinates, \( \nabla f = f_x \hat{i} + f_y \hat{j} \).

3. Divergence = \( \nabla \cdot \vec{F} \). It turns vector fields \( \vec{F} \) into scalar fields/functions \( \nabla \cdot \vec{F} = \text{div} \vec{F} \). It is the local indicator of the flux out of a region. (“Divergence = the density of flux.”) Positive divergence means source. Negative divergence means sink.
   
   In coordinates, \( \nabla \cdot (M \hat{i} + N \hat{j}) = M_x + N_y \).
   
   Green’s theorem (normal form): \( \int_C \vec{F} \cdot \vec{n} \, dS = \int_R \nabla \cdot \vec{F} \, dA \)

   Coordinate version of normal Green’s theorem:
   \[
   \vec{F} = M \hat{i} + N \hat{j} \\
   \vec{n} \, dS = dy \hat{i} - dx \hat{j} \\
   \int_C -N \, dx + M \, dy = \int_R (M_x + N_y) \, dA
   \]

4. Curl = \( \nabla \times \vec{F} \). It turns vector fields \( \vec{F} \) into vector fields \( \nabla \times \vec{F} = (\text{curl} \vec{F}) \hat{k} \). Its direction is the axis of rotation. (Since we are dealing with 2D functions for now, the axis is the \( z \)-axis). Its magnitude is twice the local angular velocity around that axis associated with the vector field \( \vec{F} \). Positive curl means rotating counterclockwise. Negative curl means rotating clockwise.
   
   In coordinates, \( \nabla \times (M \hat{i} + N \hat{j}) = (N_x - M_y) \hat{k} \).
   
   Green’s theorem (tangential form): \( \int_C \vec{F} \cdot \vec{T} \, dS = \int_R (\nabla \times \vec{F}) \cdot \hat{k} \, dA \)

   Coordinate version of tangential Green’s theorem:
   \[
   \vec{F} = M \hat{i} + N \hat{j} \\
   \vec{T} \, dS = dx \hat{i} + dy \hat{j} \\
   \int_C M \, dx + N \, dy = \int_R (N_x - M_y) \, dA
   \]
2 Problems

1. Let \( \vec{F} \) be the vector field
\[
\vec{F} = \sin(y^3) \hat{i} + \cos^3(x) \hat{j}
\]
and \( C_1, C_2 \) be the curves below with prescribed orientation. If \( F_1 \) is the flux of \( \vec{F} \) across \( C_1 \) and \( F_2 \) is the flux of \( \vec{F} \) across \( C_2 \), then find \( F_2 - F_1 \).

![Diagram of curves \( C_1, C_2 \)]

2. Let \( \vec{F} \) be the vector field
\[
\vec{F} = \frac{(x - 2y) \hat{i} + (2x + y) \hat{j}}{x^2 + y^2}
\]
and \( C \) be the curve below with prescribed orientation. Compute the work done by \( \vec{F} \) along \( C \).

![Diagram of curve \( C \)]

3. Let \( \vec{F} \) be the vector field
\[
\vec{F} = \frac{x \hat{i} + y \hat{j}}{x^2 + y^2}
\]
Let \( C \) be the circle centered at \((a, 0)\) with radius 2. Compute the flux of \( \vec{F} \) through \( C \).

4. Let \( \vec{F} \) be the vector field
\[
\vec{F} = \frac{-y \hat{i} + x \hat{j}}{x^2 + y^2}
\]
Compute the work done by \( \vec{F} \) across the curve \( C \), drawn below with prescribed orientation.

![Diagram of circle with 3 curves]