Recitation 13: Applications of double integrals
18.02 Section R21
October 23, 2017

1 Lecture review

1.1 Average values
Think “average = sum/total.” When \( f \) represents a density, then the average value computes the average density.

<table>
<thead>
<tr>
<th>Single variable ( y = f(x) )</th>
<th>Average value</th>
<th>Numerator (“Sum”)</th>
<th>Denominator (“Total”)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int_a^b f(x) , dx )</td>
<td>( \frac{\int_a^b f(x) , dx}{b-a} )</td>
<td>( \int_a^b f(x) , dx = \text{mass of rod occupying} \ [a, b] ) with density ( f(x) )</td>
<td>( b - a = \int_a^b 1 , dx = \text{Length}(I) )</td>
</tr>
</tbody>
</table>

| Two variable \( z = f(x,y) \) | \( \frac{\int_R f(x,y) \, dA}{\int_R 1 \, dA} \) | \( \int_R f(x,y) \, dA = \text{mass of 2D object occupying} \ R \) with density \( f(x,y) \) | \( \int_R 1 \, dA = \text{Area}(R) \) |

1.2 Mass, center of mass, centroid

1. (Mass)
As written in the second column of the second row of the previous section’s table,
\[
\int_R \rho(x,y) \, dA = \text{mass of 2D object occupying a region } R \text{ with density } \rho(x,y)
\]

2. (Coordinates for center of mass)
The center of mass of a 2D object occupying a region \( R \) with density \( \rho(x,y) \) is given by a point \((x_{CM}, y_{CM})\).
The formula is
\[
x_{CM} = \frac{\int_R x \rho(x,y) \, dA}{\int_R \rho(x,y) \, dA} \\
y_{CM} = \frac{\int_R y \rho(x,y) \, dA}{\int_R \rho(x,y) \, dA}
\]

3. (Coordinates for centroid AKA “geometric center of mass”)
This is a special case of the center of mass where \( \rho(x,y) = 1 \). The centroid (also known as the “geometric center of mass”) of a 2D region \( R \), denoted \((\overline{x}, \overline{y})\) is
\[
\overline{x} = \text{average value of the function } f(x,y) = x \text{ over } R \\
\overline{y} = \text{average value of the function } f(x,y) = y \text{ over } R
\]
\[
(\overline{x}, \overline{y}) = \left( \frac{\int_R x \, dA}{\int_R 1 \, dA}, \frac{\int_R y \, dA}{\int_R 1 \, dA} \right)
\]

4. Important note: Often symmetry arguments can let you skip the integrals to compute center of mass.

1.3 Moment of inertia

1. The moment of inertia takes in an object, an axis (a line), and outputs a value which represents “the object’s resistance to angular accelerations around the axis.”

2. For a point of mass \( m \) located a distance \( d \) from the axis, the moment of inertia is
\[
I = md^2.
\]

3. For a 2D region \( R \) of density \( \rho(x,y) \) and an axis whose distance from the point \((x,y)\) is \( d(x,y) \), the moment of inertia is
\[
I = \int_R \rho(x,y)d(x,y)^2 \, dA
\]
Note the parallels between this formula and the previous one.
2 Problems

2.1 A useful integral computation from lecture

For integers $m$ and $n$,

$$
\int_{m\pi/2}^{n\pi/2} \sin^2 \theta \, d\theta = \int_{m\pi/2}^{n\pi/2} \cos^2 \theta \, d\theta = \frac{1}{2} \int_{m\pi/2}^{n\pi/2} (\sin^2 \theta + \cos^2 \theta) \, d\theta = \frac{1}{2} \int_{m\pi/2}^{n\pi/2} 1 \, d\theta = \frac{(n - m)\pi}{4}.
$$

Warning: This only works when $m$ and $n$ are integers. Do not substitute other values for $m$ and $n$; the result will not be correct.

2.2 Mass, center of mass, centroid

Compute the mass, center of mass, and centroid for the following objects.

1. The interior of the cardoid $r(\theta) = 2a(1 - \cos \theta)$ with density $1/r$.
2. The plane region defined by $r < a$, $0 < \theta < 2\alpha$ with density 1.
3. The square with vertices $(0, 0)$, $(0, a)$, $(a, a)$, $(a, 0)$ with density $x + y$.
4. A triangular plate bounded by the lines $x = 0$, $y = 0$, and $x + y = a$ with density $x$.

2.3 Moment of inertia

Compute the moment of inertia for the following objects about the given axes.

1. The plane region defined by $r < a$, $0 < \theta < 2\alpha$ and the $x$-axis.
2. The plane region defined by $r < a$, $0 < \theta < 2\alpha$ and the $z$-axis.
3. The part of the disk of radius $a$ (centered at the origin) lying in the first quadrant with density $\sqrt{x^2 + y^2}$ and the $y$-axis.
4. One loop of the lemniscate $r^2 = a^2 \cos(2\theta)$ and the $z$-axis.