Recitation 12: Polar coordinates, average values
18.02 Section R21
October 18, 2017

1 Lecture review

1.1 Review of multiple integrals

1. (Usual 1D integration) Area under the curve \( y = f(x) \) over the interval \( I = [a, b] \) is

\[
\int_a^b f(x) \, dx = \int_a^b \int_0^1 dy \, dx.
\]

Note: when \( f(x) = 1 \), this gives the length of the interval \( I \).

2. (Vertical strips) Volume under the surface \( z = f(x, y) \) over the region \( R = \{(x, y) : a \leq x \leq b, y_B(x) \leq y \leq y_T(x)\} \) is

\[
\int_a^b \int_{y_B(x)}^{y_T(x)} f(x, y) \, dy \, dx = \int_a^b \int_{y_B(x)}^{y_T(x)} f(x, y) \, dy \, dx.
\]

As \( x \) is going from \( a \) to \( b \), \( y \) is going from \( y_B(x) \) to \( y_T(x) \).

3. (Horizontal strips) Volume under the surface \( z = f(x, y) \) over the region \( R = \{(x, y) : c \leq y \leq d, x_L(y) \leq x \leq x_R(y)\} \) is

\[
\int_c^d \int_{x_L(y)}^{x_R(y)} f(x, y) \, dx \, dy = \int_c^d \int_{x_L(y)}^{x_R(y)} f(x, y) \, dx \, dy.
\]

As \( y \) is going from \( c \) to \( d \), \( x \) is going from \( x_L(y) \) to \( x_R(y) \).

Important: If integrating in one order does not work, try the other.

1.2 Polar coordinates

1. Conversion formulas:

\[
x = r \cos \theta \\
y = r \sin \theta \\
r = \sqrt{x^2 + y^2} \\
\theta = \tan^{-1}(y/x).
\]

2. Conversion of integral from rectangular to polar:

\[
\int \int_{x,y \text{ bounds}} f(x, y) \, dx \, dy = \int \int_{r, \theta \text{ bounds}} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.
\]

Warning: Do not forget the extra \( "r" \) that appears! Sometimes it makes the difference between being able to evaluate the integral and not.

3. Analogous to vertical and horizontal strips, now one has “radial strips” and “azimuthal” strips, depending on the order of integration.

(a) Radial: \( \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2(\theta)} \ldots \, dr \, d\theta \). That is, as \( \theta \) goes from \( \theta_1 \) to \( \theta_2 \), \( r \) is going from \( r_1(\theta) \) to \( r_2(\theta) \).

This is usually the order that is easiest to compute.

(b) Azimuthal: \( \int_{r_1}^{r_2} \int_{\theta_1(r)}^{\theta_2(r)} \ldots \, d\theta \, dr \). That is, as \( r \) goes from \( r_1 \) to \( r_2 \), \( \theta \) is going from \( \theta_1(r) \) to \( \theta_2(r) \).

It is uncommon for this order to be easier to compute.

1.3 Average values

Think “average = sum/total.” When \( f \) represents a density, then the average value computes the average density.

<table>
<thead>
<tr>
<th>Single variable ( y = f(x) )</th>
<th>Average value ( \frac{\int_a^b f(x) , dx}{b - a} )</th>
<th>Numerator (“Sum”) ( \int_a^b f(x) , dx = \text{mass of rod occupying } I = [a, b] \text{ with density } f(x) )</th>
<th>Denominator (“Total”) ( b - a = \int_a^b 1 , dx = \text{Length}(I) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two variable ( z = f(x, y) )</td>
<td>Average value ( \frac{\int_R f(x, y) , dA}{\int_R 1 , dA} )</td>
<td>Numerator (“Sum”) ( \int_R f(x, y) , dA = \text{mass of 2D object occupying } R \text{ with density } f(x, y) )</td>
<td>Denominator (“Total”) ( \int_R 1 , dA = \text{Area}(R) )</td>
</tr>
</tbody>
</table>
2 Problems

2.1 Double integrals

1. For the following problems, compute \( \int_R f(x, y) \, dA \) for the given \( f \) and \( R \). The choice of integration order will be important.

(a) \( f(x, y) = ye^{x^3}, R \) is the interior of the triangle with vertices at \( (0, 0), (1, 0), (1, 1) \).

\[
\int_0^1 \int_0^x ye^{x^3} \, dy \, dx = \frac{1}{2} \int_0^1 x^2 e^{x^3} \, dx
\]

\[
\left( \int_0^1 \int_x^1 ye^{x^3} \, dy \, dx \right) = \frac{1}{2} \left[ \frac{1}{3} e^{x^3} \right]_0^1 = \frac{1}{6} (e - 1).
\]

(b) \( f(x, y) = e^x / (y \ln x), R \) is the interior of the triangle with vertices at \( (1, 1), (2, 1), (2, 2) \).

\[
\int_1^2 \int_1^x \frac{e^x}{y \ln x} \, dy \, dx = \int_1^2 e^x \, dx = e^2 - e
\]

\[
\left( \int_1^2 \int_1^x \frac{e^x}{y \ln x} \, dy \, dx \right) = \int_1^2 \frac{e^x}{y \ln x} \, dx
\]

\[
= \int_1^2 \sin u \, du \left( u = \sqrt{x^2 + 2} \right)
\]

\[
= \cos 1 - \cos \sqrt{2}
\]

(c) \( f(x, y) = \cos \left( \sqrt{x^2 + (x/y)^2} \right), R \) is the interior of the triangle with vertices at \( (0, 0), (0, 1), (1, 1) \).

\[
\int_0^1 \int_0^y \cos \sqrt{x^2 + (x/y)^2} \, dx \, dy = \int_0^1 \left[ \sin \sqrt{x^2 + y^2} \right]_0^y \, dy
\]

\[
\left( \int_0^1 \int_0^x \cos \sqrt{x^2 + (x/y)^2} \, dx \, dy \right) = \int_0^1 \frac{y \sin \sqrt{4y^2 + 1}}{y^2 + 1} \, dy
\]

\[
= \int_1^{\sqrt{2}} \sin u \, du \left( u = \sqrt{x^2 + 2} \right)
\]

\[
= \cos 1 - \cos \sqrt{2}
\]

2. (3B-2) Use polar coordinates to compute \( \int_R f(x, y) \, dA \) for the given \( f \) and \( R \).

(a) \( f(x, y) = 1/\sqrt{x^2 + y^2}, R \) is the region inside the first-quadrant loop of \( r = \sin(2\theta) \).

\[
\int_0^{\pi/2} \int_0^{\sin 2\theta} \frac{1}{r} (r \, dr \, d\theta) = \int_0^{\pi/2} \sin 2\theta \, d\theta
\]

\[
= \left[ -\frac{1}{2} \cos 2\theta \right]_0^{\pi/2} = -\frac{1}{2} (\cos \pi - \cos 0)
\]

\[
= 1.
\]
(b) \( f(x, y) = 1/(1 + x^2 + y^2) \), \( R \) is the first-quadrant portion of the interior of \( x^2 + y^2 = a^2 \).

\[
\int_0^\pi \int_0^a \frac{1}{1+r^2} \, (r \, dr \, d\theta) = \int_0^{\frac{\pi}{2}} \left[ \frac{1}{2} \ln(1+r^2) \right]_0^a \, d\theta \\
= (\frac{\pi}{2}) \cdot (\frac{1}{2} \ln(1+a^2)) \\
= \frac{\pi}{4} \ln(1+a^2).
\]

(c) \( f(x, y) = y^2/x^2 \), \( R \) is the interior of the triangle with vertices at \((0, 0)\), \((0, 1)\), \((1, 1)\).

\[
\int_0^\pi \int_0^{\sec \theta} \tan^2 \theta \, (r \, dr \, d\theta) \\
= \frac{1}{2} \int_0^{\pi/4} \sec \theta \tan^2 \theta \, d\theta \\
= \frac{1}{2} \left[ \frac{1}{2} \tan^2 \theta \right]_0^{\pi/4} = \sqrt{2}.
\]

(Rectangular: \( \int_0^1 \int_0^y \frac{y}{x^2} \, dx \, dy = \int_0^{\pi/4} \left[ \frac{1}{2} \frac{y^2}{x^2} \right]_0^1 \, dx \))

\[
= \frac{1}{2} \int_0^1 \, dx = \frac{1}{2}.
\]

(d) \( f(x, y) = 1/\sqrt{1 - x^2 - y^2} \), \( R \) is the right half-disk of radius \( \frac{1}{2} \) centered at \((0, \frac{1}{2})\).

\[
\int_0^{\pi/2} \int_0^{1/\sqrt{1-r^2}} \frac{1}{1-r^2} \, (r \, dr \, d\theta) \\
= \int_0^{\pi/2} \left[ -\sqrt{1-r^2} \right]_0^{1/\sqrt{1-r^2}} \, d\theta \\
= \int_0^{\pi/2} (1-\cos \theta) \, d\theta \\
= \left[ \theta - \sin \theta \right]_0^{\pi/2} \\
= \pi/2 - 1.
\]

3. Evaluate the following double integrals using any method.

(a) \( \int_1^2 \int_0^x (x^2 + y^2)^{-3/2} \, dy \, dx \)

\[
x = r \sin \theta \quad 0 \leq \theta \leq \pi/4 \\
y = r \cos \theta \quad \sec \theta \leq r \leq 2 \sec \theta \\
\int_0^{\pi/4} \int_0^{2 \sec \theta} r^{-3} \, (r \, dr \, d\theta) \\
= \int_0^{\pi/4} \left[ -r^{-1} \right]_0^{2 \sec \theta} \, d\theta \\
= \int_0^{\pi/4} \frac{1}{2} \cos \theta \, d\theta \\
= \left[ \frac{1}{2} \sin \theta \right]_0^{\pi/4} \\
= \sqrt{2}/4.
\]
(b) \[ \int_{0}^{1} \int_{y^{1/3}}^{1} \frac{1}{\sqrt{1+x^2}} \, dx \, dy \]

\[ 0 \leq y \leq 1 \quad 0 \leq x \leq 1 \]

\[ y^{\frac{1}{3}} \leq x \leq 1 \quad 0 \leq y \leq x^{3} \]

\[ = \int_{0}^{1} \int_{0}^{x^{3}} \frac{1}{\sqrt{1+x^2}} \, dy \, dx \]

\[ = \int_{0}^{1} \int_{0}^{x^{3}} \frac{x^{3}}{\sqrt{1+x^2}} \, dx \, \frac{dx}{2} \quad u = 1 + x^2 \quad du = 2x \, dx \]

\[ = \int_{0}^{1} \left[ \frac{1}{3} u^{3/2} - \frac{1}{2} u^{1/2} \right] \, du \]

\[ = \frac{1}{3} (2 - 1) \]

(c) \[ \int_{0}^{1} \int_{0}^{\sqrt{1-y^2}} \sin(x^2 + y^2) \, dx \, dy \]

\[ x = r \cos \theta \quad y = r \sin \theta \]

\[ 0 \leq r \leq 1 \quad 0 \leq \theta \leq \pi/2 \]

\[ = \int_{0}^{\pi/2} \int_{0}^{1} \sin(r^2) \, r \, dr \, d\theta \]

\[ = \frac{\pi}{4} \left( 1 - \cos 1 \right) \]

(d) \[ \int_{0}^{4} \int_{0}^{\sqrt{y}} \frac{y \exp(x^2)}{x^3} \, dx \, dy \]

\[ 0 \leq y \leq 4 \quad 0 \leq x \leq 2 \]

\[ r \leq 2 \quad 0 \leq y \leq x^2 \]

\[ = \int_{0}^{4} \int_{0}^{x^2} \frac{y \exp(x^2)}{x^3} \, dy \, dx \]

\[ = \int_{0}^{4} \int_{0}^{x^2} \left[ \frac{1}{2} y^2 \right] \, dx \, dy \]

\[ = \int_{0}^{4} \left[ \frac{1}{4} \exp(x^2) \right] \, dx \]

\[ = \frac{1}{4} \left( e^{4} - 1 \right) \]

(e) \[ \int_{0}^{2} \int_{0}^{\sqrt{4-x^2}} (x^2 + y^2)^{3/2} \, dy \, dx \]

\[ 0 \leq x \leq 2 \quad 0 \leq r \leq 2 \]

\[ 0 \leq y \leq \sqrt{4-x^2} \quad 0 \leq \theta \leq \pi/2 \]

\[ = \int_{0}^{\pi/2} \int_{0}^{2} r^3 \sin(r \, d\theta) \, r \, dr \, d\theta \]

\[ = \int_{0}^{\pi/2} \left[ \frac{1}{6} r^6 \right] \, d\theta \]

\[ = \frac{\pi}{2} \cdot \left( \frac{32}{6} \right) \]

\[ = 16\pi/15. \]
4. (3A-4b) Find the volume of the solid lying over the finite region \( R \) in the first quadrant between the graphs of \( x \) and \( x^2 \), and underneath the graph of \( z = xy \).

\[
\int_0^1 \int_{x^2}^x (xy) \, dy \, dx = \int_0^1 \left[ \frac{1}{2} x y^2 \right]_{x^2}^x \, dx \\
= \frac{1}{2} \int_0^1 (x^3 - x^5) \, dx \\
= \frac{1}{2} \left( \frac{1}{4} - \frac{1}{6} \right) \\
= \frac{1}{24}.
\]

5. Find the volume above the \( xy \)-plane and bounded by the cylinder \( x^2 + y^2 = a^2 \) and the paraboloid \( az = x^2 + y^2 \).

\[
\int_0^{2\pi} \int_0^a \left( \frac{r^2}{a} \right) (r \, dr \, d\theta) \\
= \int_0^{2\pi} \left[ \frac{1}{4} \frac{r^4}{a} \right]_0^a \, d\theta \\
= 2\pi \cdot \frac{1}{4} a^3 \\
= \pi a^3/2.
\]

2.2 Average values

1. Find the mass, area, and average density of the following.

(a) the square with vertices \((0, 0), (0, a), (a, 0), \text{ and } (a, a)\), density is given by \( \delta(x, y) = x + y \).

\begin{align*}
\text{mass:} & \quad \int_0^a \int_0^a (x+y) \, dx \, dy = \int_0^a \frac{1}{2} a^2 + ay \, dy = \frac{1}{2} a^3 + \frac{1}{2} a^2 = a^3 \\
\text{area:} & \quad a^2 \\
\text{average density:} & \quad a^3/a^2 = a.
\end{align*}

(b) the region bounded by the parabolas \( y = x^2 \) and \( y = 2 - x^2 \), density is given by \( \delta(x, y) = y \).

\begin{align*}
\text{mass:} & \quad \int_{-1}^1 \int_{x^2}^{2-x^2} y \, dy \, dx = \int_{-1}^1 \frac{1}{2} \left( (2-x^2)^2 - (x^2)^2 \right) \, dx = \int_{-1}^1 2-2x^2 \, dx = \frac{8}{3} \\
\text{area:} & \quad \int_{-1}^1 \int_{x^2}^{2-x^2} \, dy \, dx = \int_{-1}^1 (2-2x^2) \, dx = 8/3 \\
\text{average density:} & \quad 1.
\end{align*}
(c) the region inside the circle $r = 2 \sin \theta$ and outside the circle $r = 1$, density is given by $\delta(x, y) = y$.

\[ \begin{align*}
\text{mass: } & \int_{\pi/6}^{5\pi/6} \int_{1}^{2\sin \theta} (r \sin \theta) \, r \, dr \, d\theta = \int_{n/6}^{5n/6} \sin \theta \left[ \frac{1}{3} r^3 \right]_1^{2\sin \theta} \, d\theta = \frac{1}{3} \int_{n/6}^{5n/6} 8 \sin \theta \sin \theta \, d\theta = (\ldots) = \left( \frac{8\pi + 3\sqrt{3}}{12} \right) \frac{\pi}{6} = \frac{8\pi + 3\sqrt{3}}{4n/6 + \sqrt{3}}.
\end{align*} \]

area: \[ \int_{\pi/6}^{5\pi/6} \int_{1}^{2\sin \theta} r \, dr \, d\theta = \int_{n/6}^{5n/6} \frac{1}{2} \left( 4 \sin^2 \theta - 1 \right) \, d\theta = \frac{1}{2} \left[ \theta - \sin \theta \right]_{\pi/6}^{5\pi/6} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}. \]

average density: \[ \frac{\frac{8\pi + 3\sqrt{3}}{12}}{\frac{\pi}{3} + \frac{\sqrt{3}}{2}} = \frac{8\pi + 3\sqrt{3}}{4n/6 + \sqrt{3}}. \]

(d) the region bounded by the cardoid with polar equation $r = 1 + \cos(\theta)$, density is given by $\delta(r, \theta) = r$.

\[ \begin{align*}
\text{mass: } & \int_{0}^{2\pi} \int_{0}^{1+\cos \theta} (r) \, r \, dr \, d\theta = \int_{0}^{2\pi} \frac{1}{3} \left[ r^3 \right]_0^{1+\cos \theta} \, d\theta = \frac{1}{3} \int_{0}^{2\pi} (1 + \cos^3 \theta) \, d\theta = (\ldots) = \frac{5\pi}{3}.
\end{align*} \]

area: \[ \int_{0}^{2\pi} \int_{0}^{1+\cos \theta} (1) \, r \, dr \, d\theta = \int_{0}^{2\pi} \frac{1}{2} \left[ r^2 \right]_0^{1+\cos \theta} \, d\theta = \frac{1}{2} \int_{0}^{2\pi} (1 + \cos \theta)^2 \, d\theta = (\ldots) = \frac{3\pi}{2}. \]

average density: \[ \left( \frac{5\pi/3}{3\pi/2} \right) = \frac{10}{9}. \]