1 Lecture review

1.1 Chain rule

<table>
<thead>
<tr>
<th>Single variable chain rule</th>
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<tbody>
<tr>
<td>If ( y = f(g(t)) ), then ( \frac{dy}{dt} = f'(g(t))g'(t) )</td>
</tr>
<tr>
<td>If ( y = f(x) ) and ( x = g(t) ), then ( \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} ).</td>
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</tbody>
</table>

The analogue of the right hand side is the easiest to write down for multivariable functions:

<table>
<thead>
<tr>
<th>Some examples of multivariable chain rule</th>
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<tr>
<td>If ( w = w(x, y) ), ( x = x(t) ), ( y = y(t) ), then ( \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} )</td>
</tr>
<tr>
<td>If ( w = w(x, y) ), ( x = x(u, v) ), ( y = y(u, v) ), then ( \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} ), ( \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} ).</td>
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1.2 Double integration

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<th>Single variable</th>
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<tr>
<td>Interval ( I = [a, b] )</td>
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<tr>
<td>Area under the curve ( y = f(x) ) is ( \int_I f(x) , dx = \int_a^b f(x) , dx ).</td>
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<th>Two variable</th>
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<td>Region ( R )</td>
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<tr>
<td>Volume under the surface ( z = f(x, y) ) is ( \int_R f(x, y) , dA ).</td>
</tr>
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</table>

1. If the region \( R \) is the rectangular box given by \( a \leq x \leq b, c \leq y \leq d \), then

\[
\int_R f(x, y) \, dA = \int_a^b \left( \int_c^d f(x, y) \, dy \right) dx = \int_c^d \left( \int_a^b f(x, y) \, dx \right) dy
\]

(a) Swapping the order of integration is allowed, but make sure to also swap the bounds.

2. If the region is not rectangular, choose the bounds of integration more carefully. You can either

(i) find \( y_B(x) \), \( y_T(x) \) such that \( R \) is given by \( a \leq x \leq b, y_B(x) \leq y \leq y_T(x) \) so that

\[
\int_R f(x, y) \, dA = \int_a^b \left( \int_{y_B(x)}^{y_T(x)} f(x, y) \, dy \right) dx
\]

(Vertical Strips)

(ii) find \( x_L(y) \), \( x_R(y) \) such that \( R \) is given by \( c \leq y \leq d, x_L(y) \leq x \leq x_R(y) \) so that

\[
\int_R f(x, y) \, dA = \int_c^d \left( \int_{x_L(y)}^{x_R(y)} f(x, y) \, dx \right) dy
\]

(Horizontal Strips)

(a) Sometimes one of these is very hard/impossible while the other is more tractable; if you get stuck doing vertical strips, make sure to try horizontal (and vice versa).
2 Problems

2.1 Chain rule

1. Let \( f(x, y, z) = x - yz \) be a function of three variables. Let

\[
g(t) = f(\sin(t), \cos(t), \tan(t))
\]

Compute \( g'(t) \) in two ways: (i) using the multivariable chain rule and (ii) computing \( g(t) \) first, and differentiating that expression.

2. A shark moves underwater which has a temperature of \( T(x, y, z) = \exp(-4x^2 + y^2 + 3z^2) \) at the point \((x, y, z)\). The shark follows a helical path given by \( x = \cos(t), y = 2\sin(t), z = t \). Compute the rate of change of temperature that the shark experiences.

2.2 Double integration

1. Compute the area between the curves \( y = x \) and \( y = x^2 \) using (i) vertical strips and (ii) horizontal strips. Check that your answers match.

2. Compute the area between the curves \( y = (e - 1)x \) and \( y = e^x - 1 \) using (i) vertical strips and (ii) horizontal strips. Check that your answers match.

3. For the following problems, compute \( \int_R f(x, y) \, dA \) for the given \( f \) and \( R \). The choice of integration order will be important.

   (a) \( f(x, y) = ye^{x^2} \), \( R \) is the interior of the triangle with vertices at \( (0, 0), (1, 0), (1, 1) \).

   (b) \( f(x, y) = e^x/(y \ln x) \), \( R \) is the interior of the triangle with vertices at \( (1, 1), (2, 1), (2, 2) \).

   (c) \( f(x, y) = \cos(\sqrt{x^2 + (x/y)^2}) \), \( R \) is the interior of the triangle with vertices at \( (0, 0), (0, 1), (1, 1) \).

4. Find the volume of the following.

   (a) A pyramid lying in the \( x, y, z \geq 0 \) octant and bounded by \( ax + by + cz \leq d \). (Assume that \( a, b, c, d \geq 0 \).)

   (b) (3A-4b) The solid lying over the finite region \( R \) in the first quadrant between the graphs of \( x \) and \( x^2 \), and underneath the graph of \( z = xy \).