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1 3D change of variables (cylindrical, spherical)

1.1 Lecture review

1. Given a change of coordinates \( x = x(u, v, w), \ y = y(u, v, w) \) and \( z = z(u, v, w) \), one has

\[
\iiint_D f(x, y, z) \, dV = \iiint_{T(D)} F(u, v, w) |J| \, du \, dv \, dw
\]

where

\[
F(u, v, w) = f(x(u, v, w), y(u, v, w), z(u, v, w)) \quad J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}
\]

One could also compute \( J \) by computing \( J^{-1} \) first as follows.

\[
J^{-1} = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \det \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}
\]

2. Cylindrical coordinates are good for problems with a line of symmetry, or with volumes that are pieces of a cylinder. Take \( x = r \cos \theta, \ y = r \sin \theta, \ z = z \). Then \( J = r \). (So \( dV = rdz \, dr \, d\theta \).)

3. Spherical coordinates are good for problems with a point of symmetry, or with volumes that are pieces of a sphere. Take \( x = \rho \cos \theta \sin \varphi, \ y = \rho \sin \theta \sin \varphi, \ z = \rho \cos \varphi \). Then \( J = \rho^2 \sin \varphi \). (So \( dV = \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \).)

1.2 Problems

1. Let \( D \) be the region that lies between the downward facing paraboloid \( z = 3 - x^2 - y^2 \) and the cone \( z = 2\sqrt{x^2 + y^2} \).

   (a) Set up the integral in cylindrical coordinates for the centroid of \( D \).
   (b) Set up the integral in cylindrical coordinates for the moment of inertia of \( D \) about the \( x \)-axis.

2. Let \( H \) be a solid hemisphere of radius \( a \) in the region \( z \geq 0 \) whose density function is \( \delta(x, y, z) = bz \).

   (a) Set up the integral in spherical coordinates for the center of mass of \( H \).
   (b) Set up the integral in spherical coordinates for the moment of inertia of \( H \) about the \( z \)-axis.

3. Let \( D \) be the region in the first octant \( (x, y, z \geq 0) \) below the surface \( \sqrt{x} + \sqrt{y} + \sqrt{z} = 1 \). Use the 3D change of coordinates \( x = u^2, \ y = v^2, \ z = w^2 \) to find the volume of \( D \).
2 Gravitational attraction

2.1 Lecture review

1. The gravitational attraction felt by two point masses is \( F = G M_1 M_2 / r^2 \).

2. The gravitational force felt by a point of mass \( M \) at the origin by a volume \( V \) of density \( \delta \) is

\[
\vec{F} = \iiint_{V} \left( \frac{G M \delta(x, y, z)}{\rho^2} \right) \hat{\rho} \, dV
\]

3. Use symmetry extensively to simplify calculation. If \( V \) has an axis of symmetry \( L \) and the point is on \( L \), then the gravitational attraction must be a vector in the direction of \( L \).

2.2 Problems

1. Let \( D \) be the “ice cream cone” domain above the cone \( z = \sqrt{3(x^2 + y^2)} \) and inside the sphere centered at the origin of radius 2. Suppose that \( D \) has density \( \delta = x^2 + y^2 + z^2 \). Compute the gravitational attraction that \( D \) exerts on a point at the origin with unit mass.

2. Let \( D \) be the domain given in spherical coordinates by \( 0 \leq \rho \leq 1 + \cos \varphi \) with density \( \delta = \cos \varphi \). Compute the gravitational attraction that \( D \) exerts on a point at the origin with unit mass.

3. Let \( D \) be the solid upper hemisphere of radius \( a \) centered at the origin with density \( \delta = \sqrt{x^2 + y^2} \). Compute the gravitational attraction that \( D \) exerts on a point at the origin with unit mass.
3 Surface integrals, flux

3.1 Lecture review

1. Recall that for a surface \( z = f(x, y) \) we have

\[
\begin{align*}
  g(x, y, z) &= z - f(x, y) \\
  \nabla g &= -f_x \hat{i} - f_y \hat{j} + \hat{k} \text{ is normal to } S \\
  \hat{n} &= -f_x \hat{i} - f_y \hat{j} + \hat{k} \\
  dS &= |\nabla g| dA = \sqrt{f_x^2 + f_y^2 + 1} dA \\
  d\vec{S} &= \hat{n} dS = \nabla g dA = (-f_x \hat{i} - f_y \hat{j} + \hat{k}) dA \\
  \iiint_S \vec{F} \cdot d\vec{S} &= \iiint_R \vec{F} \cdot (-f_x \hat{i} - f_y \hat{j} + \hat{k}) dA
\end{align*}
\]

2. (Formulas in cylindrical coordinates)

   If \( S \) is part of a cylinder of radius \( a \) with a central \( z \)-axis, then

\[
\hat{n} = \hat{r} = \frac{x\hat{i} + y\hat{j}}{a} = \cos \theta \hat{i} + \sin \theta \hat{j}
\]

3. (Formulas in spherical coordinates)

   If \( S \) is part of a sphere of radius \( a \) centered at the origin, then

\[
\hat{n} = \hat{\rho} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} = \sin \varphi \cos \theta \hat{i} + \sin \varphi \sin \theta \hat{j} + \cos \varphi \hat{k}
\]

3.2 Problems

1. Let \( S \) be the surface \( z = \frac{2}{3}(x^{3/2} + y^{3/2}) \) lying above the shadow region \( R = \{0 \leq x, y \leq 1\} \). Compute the \( x \)-coordinate of the centroid of \( S \).

2. Let \( S \) be the portion of the cylinder \( x^2 + y^2 = 1 \) in the octant \( x, y, z \geq 0 \) that lies below \( z = 1 \). Compute the flux of \( \vec{F} = (x^3z^2 + y^2z)\hat{i} + (x^2yz^2 - xyz)\hat{j} + (xz^4 - y^5)\hat{k} \) through \( S \).

3. Let \( S \) be the portion of the sphere of radius 2 centered at the origin between \( z = 0 \) and \( z = \sqrt{3} \). Compute the flux of \( \vec{F} = xz\hat{i}/\sqrt{4 - z^2} \) through \( S \).
4 Divergence theorem

4.1 Lecture review

1. Recall the 2D version: for a closed, simple, piecewise smooth, positively oriented curve \( C \) and a vector field \( \mathbf{F} \) we have

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \nabla \cdot \mathbf{F} \, dA
\]

(We also called this the normal form of Green’s theorem.)

2. The 3D version is as follows. Let \( S \) be a closed piecewise smooth surface bounding a space region \( D \) with outward unit normal \( \mathbf{n} \). Then for a vector field \( \mathbf{F} \) we have

\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_D \nabla \cdot \mathbf{F} \, dV
\]

3. The divergence of the gravitational field generated by a point mass at \((0,0,0)\) is zero everywhere except for \((0,0,0)\). Hence the flux of this gravitational field is zero for any closed surface \( S \) whose interior does not contain the origin.

4.2 Problems

1. Calculate the flux of

\[
\mathbf{F} = \frac{\sin(y^2)}{1+z^2}\mathbf{i} + \frac{e^{x^2}}{\log(e^x + 1)}\mathbf{j} + (1+z^2)\mathbf{k}
\]

through the top half of the unit sphere centered at the origin, oriented upward.

2. Let \( \mathbf{F} = (3x - x^3 - xz^2)\mathbf{i} + (3y - y^3 - x^2y)\mathbf{j} + (3z - z^3 - y^2z)\mathbf{k} \).

(a) Find the closed surface \( S \) that maximizes the outward flux of \( \mathbf{F} \) through \( S \).

(b) Let \( C \) the the unit circle in the \( xy \)-plane. Find the surface \( S \) in the region \( z \geq 0 \) with boundary \( C \) which maximizes the outward flux of \( \mathbf{F} \) through \( S \).

3. Use the divergence theorem to calculate

\[
\iint_S (2x + 2y + z^2) \, dS
\]

where \( S \) is the sphere \( x^2 + y^2 + z^2 = 1 \).
5 Line integrals, conservativity, potential, curl in 3D

5.1 Lecture review

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<th>line integrals</th>
<th>curl</th>
<th>conservativity</th>
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<tr>
<td>2D</td>
<td>( \int_C M , dx + N , dy )</td>
<td>( \nabla \times (M \hat{i} + N \hat{j}) )</td>
<td>check curl is 0 in region with no holes</td>
</tr>
<tr>
<td></td>
<td>parametrize ( C ) to evaluate</td>
<td>( (N_x - M_y) \hat{k} )</td>
<td></td>
</tr>
<tr>
<td>3D</td>
<td>( \int_C M , dx + N , dy + P , dz )</td>
<td>( \nabla \times (M \hat{i} + N \hat{j} + P \hat{k}) )</td>
<td>check curl is 0 in region with no holes</td>
</tr>
<tr>
<td></td>
<td>parametrize ( C ) to evaluate</td>
<td>( (P_y - N_z) \hat{i} + (M_z - P_z) \hat{j} + (N_x - M_y) \hat{k} )</td>
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<td>3D</td>
<td>( f ) such that ( \nabla f = \vec{F} )</td>
<td>Stokes’ theorem</td>
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<td>algebraic, integration methods</td>
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The notion of conservativity is equivalent to any of the following:

\[
\vec{F} = \nabla f \iff \int_C \vec{F} \cdot d\vec{r} = 0
\]

\[\iff \int_C \vec{F} \cdot d\vec{r} \text{ depends only on endpoints} \]

\[\iff \nabla \times \vec{F} = 0 \text{ (in regions with no holes)} \]

5.2 Problems

1. Calculate the work done by \( \vec{F} = xy^2 \hat{i} + 2z^2 \hat{j} + x \hat{k} \) along the curve given by \( \vec{r}(t) = \sin(t) \hat{i} + t \hat{j} + \cos(t) \hat{k} \) from \( t = -\pi/2 \) to \( t = \pi/2 \).

2. Let \( \vec{F} \) be the vector field \( \vec{F} = (3x^2 + ayz) \hat{i} + b(xz + z^2) \hat{j} + (cxy + 2yz) \hat{k} \)
   
   (a) Find the values of \( a, b, c \) which make \( \vec{F} \) conservative.
   (b) Using the values of \( a, b, c \) found above, find a potential for \( f \) using the (i) algebraic method and the (ii) integration method. Check that your answers match.

3. Let \( \vec{F} \) be the force field \( \vec{F} = x^2 yz^2 \hat{i} + x^2 z^2 \hat{j} + 3xz^2 y^2 \hat{k} \).
   
   (a) Show that \( \vec{F} \) is conservative, and find a potential function for it.
   (b) Find the maximum and minimum values of work done by all paths lying in the unit sphere centered at the origin.
6 Stokes’ theorem

6.1 Lecture review

1. Stokes’ theorem: For a surface $S$ that is bounded, piecewise smooth, and simple with boundary curve $C$ such that $C$ and $S$ are oriented by the right hand rule, and a vector field $\vec{F}$ that is continuously differentiable on $S$, we have

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS$$

<table>
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<tr>
<th>Relationship between work and curl</th>
<th>2D version</th>
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<tr>
<td>Green’s theorem (tangential form)</td>
<td>$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\text{curl} \vec{F}) \cdot \hat{k} , dA$</td>
<td>$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} , dS$</td>
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<td>work done by $\vec{F}$ along $C$</td>
<td>work done by $\vec{F}$ along $C$</td>
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<th>Relationship between flux and div</th>
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<td>outward flux of $\vec{F}$ through $C$</td>
<td>$\iiint_D \nabla \cdot \vec{F} , dV$</td>
</tr>
<tr>
<td>outward flux of $\vec{F}$ through $C$</td>
<td>outward flux of $\vec{F}$ through $S$</td>
<td></td>
</tr>
</tbody>
</table>

2. Stokes’ theorem when $S$ is a region in the $xy$-plane = tangential form of Green’s theorem (here, $\hat{n} = \hat{k}$)

$$\text{integrand in Stokes'} = (\nabla \times (M\hat{i} + N\hat{j})) \cdot \hat{k} = N_x - M_y = \text{integrand in Green’s.}$$

3. Geometric meaning of curl $\nabla \times \vec{F}$. Qualitatively, it gives the magnitude of swirl/ angular velocity. Quantitatively, $\hat{\nu} \cdot \text{curl} \vec{F}$ is twice the angular velocity of $\vec{F}$ in the $\hat{\nu}$-direction.

4. The integral of $(\text{curl} \vec{F}) \cdot \hat{n}$ is the same for any two surfaces with same boundary. So this gives three methods for evaluating $\iint_S (\text{curl} \vec{F}) \cdot n \, dS$: (i) directly, using the formula for surface integrals, (ii) replacing $S$ with a (simpler) surface $T$ with the same boundary curve, (iii) calculating the work done by $\vec{F}$ along the boundary of $S$ (oriented via the right hand rule).

6.2 Problems

1. Verify Stokes’ theorem for $\vec{F} = (y + z)\hat{i} + (x - z)\hat{j} + (-x + y)\hat{k}$ and $C$ the curve given as the intersection of the paraboloid $z = 2 - x^2 - y^2$ and the plane $2x + 2y - z = 0$.

2. Let $C$ be the triangle with vertices at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented clockwise when viewed from above. Let $\vec{F} = (x + 2z + z^2)\hat{i} + (2x + y + x^2)\hat{j} + (2y + z + y^2)\hat{k}$. Compute the work done by $\vec{F}$ along $C$.

3. Let $\vec{F} = 3yz\hat{i} + (3x - 2z)\hat{j} + (xy - x)\hat{k}$, $R$ be the portion of the ellipsoid $x^2 + y^2/4 + z^2/9 \leq 1$ in the first octant, $S_{xy}$, $S_{yz}$, $S_{xz}$, $S_{\text{top}}$ be the boundary pieces of $R$ where the first three lie in the $xy$-, $yz$-, $xz$-planes and the fourth is the curved top surface.

   (a) Compute the outward flux of $\nabla \times \vec{F}$ across $S_{\text{top}}$ directly.

   (b) Using Stokes’ theorem, relate the above quantity to the sum of the fluxes across $S_{xy}$, $S_{yz}$, and $S_{xz}$. Compute this sum and verify that it matches your answer in the previous part.

   (c) Using Stokes’ theorem, relate the above quantities to the integral of the work done by $\vec{F}$ over the boundary. Compute this and verify that it matches your answer in the previous part.
7 Solutions

7.1 3D change of variables (cylindrical, spherical)

1. (a) The centroid only has a $z$-component by symmetry. So it is

$$\left(\iiint_D z \, dV \over \iiint_D 1 \, dV\right) \hat{k} = \left(\int_0^{2\pi} \int_0^\pi \int_0^a (\rho \cos \varphi)^2 (\rho^2 \sin \varphi) \, d\rho \, d\varphi \, d\theta\right) \hat{k} = \left(\frac{13\pi}{6} / \frac{7\pi}{6}\right) \hat{k} = \frac{13}{7} \hat{k}.$$ 

(b) The distance from the $x$-axis is $\sqrt{y^2 + z^2} = \sqrt{(r \sin \theta)^2 + z^2}$. Hence it is

$$\iiint_D ((r \sin \theta)^2 + z^2) \, dV = \int_0^{2\pi} \int_0^\pi \int_0^a (r^2 \sin^2 \theta + z^2) r \, dr \, d\theta \, d\varphi = \frac{68\pi}{15}.$$ 

Note. An earlier version of this handout wrote “the distance from the $x$-axis is $r \sin \theta$” and the subsequent computations were incorrect. This mistake was caught thanks to Isabella Montanaro and the solutions have been amended.

2. (a) The centroid only has a $z$-component by symmetry. So it is

$$\left(\iiint_D z \, dm \over \iiint_D 1 \, dm\right) \hat{k} = \left(\int_0^{2\pi} \int_0^\pi \int_0^a b(\rho \cos \varphi)^2 (\rho^2 \sin \varphi) \, d\rho \, d\varphi \, d\theta\right) \hat{k} = \left(\frac{2\pi a^5 b / 15}{\pi a^4 b / 4}\right) \hat{k} = \left(\frac{8a}{15}\right) \hat{k}.$$ 

(b) The distance from the $z$-axis is $\rho \sin \varphi$. Hence it is

$$\iiint_D (\rho \sin \varphi)^2 \, dm = \int_0^{2\pi} \int_0^\pi \int_0^a (\rho \sin \varphi)^2 (b \rho \cos \varphi) (\rho^2 \sin \varphi) \, d\rho \, d\varphi \, d\theta = \frac{\pi a^6 b}{12}.$$ 

3. The inverse Jacobian is $8uvw$. The bounds are $u, v, w \geq 0$ and $0 \leq u + v + w \leq 1$. Hence the volume is

$$\iiint_D 1 \, dV = \int_0^1 \int_0^1 \int_0^{1-u-v} 8uvw \, dw \, dv \, du = \int_0^1 \int_0^{1-u} 4uv(1-u-v)^2 \, dv \, du = \int_0^1 {u(1-u)}^4 \, du = \frac{1}{90}.$$ 

7.2 Gravitational attraction

1. By symmetry the force is only in the $z$-direction. The integrand is

$$\frac{GM\delta}{\rho^2} (\hat{\rho} \cdot \hat{k}) \hat{k} \, dV = \frac{G\rho^2}{\rho^2} (\cos \varphi) \hat{k}(\rho^2 \sin \varphi) \, d\rho \, d\varphi \, d\theta = \frac{1}{2} G\rho^2 \sin 2\varphi \hat{k} \, d\rho \, d\varphi \, d\theta$$

so the gravitational field is

$$\frac{1}{2} G \left(\int_0^{2\pi} \int_0^\pi \int_0^1 \rho^2 \sin 2\varphi \, d\varphi \, d\rho \, d\theta\right) \hat{k} = \frac{2}{3} \pi G \hat{k}$$

2. As the domain and density are symmetric about the $z$-axis, and the point is also on the $z$-axis, it follows by symmetry that the force is only in the $z$-direction. Then integrand is

$$\frac{GM\delta}{\rho^2} (\hat{\rho} \cdot \hat{k}) \hat{k} \, dV = \frac{G\cos \varphi}{\rho^2} (\cos \varphi) (\rho^2 \sin \varphi) \hat{k} \, d\rho \, d\varphi \, d\theta = G \cos^2 \varphi \sin \varphi \hat{k} \, d\rho \, d\varphi \, d\theta$$

so the gravitational field is

$$G \left(\int_0^{2\pi} \int_0^\pi \int_0^1 \cos^2 \varphi \sin \varphi \, d\varphi \, d\rho \, d\theta\right) \hat{k} = \frac{4}{3} \pi G \hat{k}.$$
3. By symmetry the gravity is in the $z$-direction, so the integrand is
\[
\frac{GM\delta}{\rho^2} (\hat{\rho} \cdot \hat{k}) \hat{k} \, dV = \frac{G (\rho \sin \varphi)}{\rho^2} (\cos \varphi) (\rho^2 \sin \varphi) \hat{k} \, d\rho \, d\varphi \, d\theta = G \rho \sin^2 \varphi \cos \varphi \hat{k} \, d\rho \, d\varphi \, d\theta
\]
so the gravitational field is
\[
G \left( \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \rho \sin^2 \varphi \cos \varphi \, d\rho \, d\varphi \, d\theta \right) \hat{k} = \frac{1}{3} \pi a^2 G \hat{k}.
\]

7.3 Surface integrals, flux

1. We compute
\[
\iint_S x \, dS = \int_0^1 \int_0^1 x \sqrt{x^2 + y^2 + 1} \, dy \, dx
\]
\[
= \int_0^1 \int_0^1 x \sqrt{x + y + 1} \, dy \, dx
\]
\[
= \frac{2}{3} \int_0^1 x (x + 2)^{3/2} - x (x + 1)^{3/2} \, dx
\]
\[
= \frac{4}{105} (9\sqrt{3} + 4\sqrt{2} - 2)
\]
\[
\iiint_S 1 \, dS = \int_0^1 \int_0^1 \sqrt{x^2 + y^2 + 1} \, dy \, dx
\]
\[
= \int_0^1 \int_0^1 \sqrt{x + y + 1} \, dy \, dx
\]
\[
= \frac{2}{3} \int_0^1 (x + 2)^{3/2} - (x + 1)^{3/2} \, dx
\]
\[
= \frac{2}{5} (9\sqrt{3} - 8\sqrt{2} + 1)
\]
so the answer is the quotient
\[
\frac{\iint_S x \, dS}{\iiint_S 1 \, dS} = \frac{2(9\sqrt{3} + 4\sqrt{2} - 2)}{21(9\sqrt{3} - 8\sqrt{2} + 1)}.
\]

2. In this instance the normal vector is $\hat{n} = x \hat{i} + y \hat{j}$ so
\[
\vec{F} \cdot \hat{n} = (x^3 z^2 + y^2 z) x + (x^2 y z^2 - xyz) y = x^2 z^2 (x^2 + y^2) = x^2 z^2 = z^2 \cos^2 \theta
\]
and hence the flux is
\[
\int_0^{\pi/2} \int_0^1 z^2 \cos^2 \theta \, dz \, d\theta = \frac{\pi}{12}.
\]

3. First we convert $\vec{F}$ to spherical coordinates:
\[
\vec{F} = \left( \frac{xz}{\sqrt{4 - z^2}} \right) \hat{i} = \left( \frac{(2 \sin \varphi \cos \theta)(2 \cos \varphi)}{\sqrt{4 - (2 \cos \varphi)^2}} \right) \hat{i} = \left( \frac{(2 \sin \varphi \cos \theta)(2 \cos \varphi)}{2 \sin \varphi} \right) \hat{i} = (2 \cos \varphi \cos \theta) \hat{i}
\]
Therefore,
\[
\mathbf{F} \cdot \hat{n} = (2 \cos \varphi \cos \theta) \hat{i} \cdot \mathbf{\hat{r}} = (2 \cos \varphi \cos \theta)(\sin \varphi \cos \theta) = 2 \sin \varphi \cos \varphi \cos^2 \theta
\]
and hence
\[
\mathbf{F} \cdot \hat{n} \, dS = (2 \sin \varphi \cos \varphi \cos^2 \theta)(2 \sin \varphi \cos \varphi \cos^2 \theta \, d\varphi \, d\theta) = 8 \sin^2 \varphi \cos \varphi \cos^2 \theta \, d\varphi \, d\theta
\]
so the flux is
\[
\int_0^{2\pi} \int_{\pi/6}^{\pi/2} (8 \sin^2 \varphi \cos \varphi \cos^2 \theta) \, d\varphi \, d\theta = \frac{7\pi}{3}.
\]

### 7.4 Divergence theorem

1. Let \( S \) be the top half of the unit sphere centered at the origin, oriented upward. Let \( T \) be the unit disk in the \( xy \)-plane centered at the origin, oriented downward. Then by the divergence theorem, the sum of the fluxes through \( S \) and \( T \) is the integral of \( \nabla \cdot \mathbf{F} \) through the solid unit upper sphere, which is
\[
\iiint \nabla \cdot \mathbf{F} \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 2z \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (2\rho \cos \varphi)(\rho^2 \sin \varphi) \, d\rho \, d\varphi \, d\theta = \pi/2.
\]
For the flux through \( T \), we note that \( \mathbf{F} \cdot \mathbf{\hat{n}} = \mathbf{\hat{F}} \cdot (-\mathbf{\hat{k}}) = -(1+z^2) = -1 \), so the flux is \(-\text{area}(T) = -\pi\). Hence the flux through \( S \) is
\[
\pi/2 - (-\pi) = 3\pi/2.
\]
2. The divergence of \( \mathbf{F} \) is \( (3 - 3x^2 - z^2) + (3 - 3y^2 - x^2) + (3 - 3z^2 - y^2) = 9 - 4(x^2 + y^2 + z^2) \).
   (a) The interior of \( S \) contains all points of positive divergence, so \( S \) is the sphere centered at the origin of radius \( 3/2 \).
   (b) Let \( T \) be the interior of \( C \), oriented upward. Then flux(\( S \)) – flux(\( T \)) equals the integral of the divergence in the region between \( S \) and \( T \), and as before, this must equal the solid upper hemisphere centered at the origin of radius \( 3/2 \).
   So \( S \) consists of two pieces: the first piece is the (surface of the) upper hemisphere centered at the origin of radius \( 3/2 \), and the second piece is the annulus in the \( xy \)-plane given by \( 1 \leq r \leq 3/2 \).
3. This integral computes the flux of the vector field \( 2\mathbf{i} + 2\mathbf{j} + \mathbf{z} \mathbf{k} \) across \( S \), so one may as well integrate the divergence of this (which is 1) across the interior of \( S \), which gives the volume. This is \( 4\pi/3 \).
   Alternatively note that the contribution from \( x \) and \( y \) is zero, since the center of mass is the origin, and then compute the surface integral by switching to spherical coordinates, which gives
\[
\int_0^{2\pi} \int_0^\pi (\cos^2 \varphi)(\sin \varphi \, d\varphi \, d\theta) = 4\pi/3.
\]
7.5 Line integrals, conservativity, potential, curl in 3D

1. Computing,

\[ \mathbf{F} \cdot d\mathbf{r} = (xy^2\mathbf{i} + 2z^2\mathbf{j} + x\mathbf{k}) \cdot (\cos(t)\mathbf{i} + \mathbf{j} - \sin(t)\mathbf{k}) \, dt \]

\[ = (t^2 \sin(t)\mathbf{i} + 2 \cos^2(t)\mathbf{j} + \sin(t)\mathbf{k}) \cdot (\cos(t)\mathbf{i} + \mathbf{j} - \sin(t)\mathbf{k}) \, dt \]

\[ = t^2 \sin(t) \cos(t) + 2 \cos^2(t) - \sin^2(t) \, dt \]

The first term is odd and as we are integrating from \(-\pi/2\) to \(\pi/2\), its contribution disappears. Hence we are left with the remaining two terms, which contribute a total of

\[ 2(\pi/2) - (\pi/2) = \pi/2. \]

2. (a) Computation yields

\[ \nabla \times \mathbf{F} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2 + ayz & b(xz + z^2) & cxy + 2yz \end{pmatrix} \]

\[ = (cx - bx + 2z - 2bz)\hat{i} + (ay - cy)\hat{j} + (bz - az)\hat{k} \]

which forces \(a = b = c = 1\) in order for this to be 0.

(b) i. Algebraic method. As \( f_x = 3x^2 + yz \) it follows that \( f = x^3 + xyz + g(y, z) \). Then \( f_y = xz + z^2 \) so this means \( g_y = z^2 \), so \( g = yz^2 + h(z) \). So \( f = x^3 + xyz + yz^2 + h(z) \). Plugging this into \( f_z = xy + 2yz \) indicates that \( h \) is a constant, so take the potential function to be \( x^3 + xyz + yz^2 \).

ii. Integration method. Let \( C \) be the straight line segment \( \mathbf{r}(t) = (tx_0\mathbf{i} + ty_0\mathbf{j} + tz_0\mathbf{k}) \) dt and \( \mathbf{F} = (3x_0^2 + y_0z_0)t^2\mathbf{i} + (x_0z_0 + z_0^2)t^2\mathbf{j} + (x_0y_0 + 2y_0z_0)t^2\mathbf{k} \) so

\[ \mathbf{F} \cdot d\mathbf{r} = ((3x_0^2 + y_0z_0)x_0 + (x_0z_0 + z_0^2)y_0 + (x_0y_0 + 2y_0z_0)z_0) \]

\[ = (x_0^3 + x_0y_0z_0 + y_0z_0^2)(3t^2 \, dt) \]

so integrating this from \( t = 0 \) to \( t = 1 \) gives \( x_0^3 + x_0y_0z_0 + y_0z_0^2 \), the same answer as before.

3. (a) The curl is

\[ \nabla \times \mathbf{F} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz^3 & xy^2z^3 & 3x^2yz^2 \end{pmatrix} \]

\[ = (3x^2z^2 - 3x^2z^2)\hat{i} + (6xyz^2 - 6xyz^2)\hat{j} + (2xz^3 - 2xz^3)\hat{k} = 0 \]

so it is conservative. The algebraic method immediately gives that a potential is \( f = x^2yz^3 \).

(b) From the fundamental theorem of calculus for line integrals, it suffices to maximize and minimize the potential function in the given region, as the work from point \( A \) to point \( B \) will be \( f(B) - f(A) \). For the maximum work, we need to maximize \( f(B) \) and minimize \( f(A) \); for the minimum, it is the reverse.

The only critical point inside is the origin. On the boundary we use Lagrange multipliers and get that the max is \( \sqrt{3}/36 \) (attained at \( x, y, z = (1/\sqrt{3}, 1/\sqrt{6}, 1/\sqrt{2}) \)) and that the min is \(-\sqrt{3}/36 \) (attained at \( x, y, z = (1/\sqrt{3}, -1/\sqrt{6}, 1/\sqrt{2}) \)).

So the maximum work is \( \sqrt{3}/18 \) and the minimum work is \(-\sqrt{3}/18 \).
7.6 Stokes’ theorem

1. Computation yields that $\nabla \times \vec{F} = 2\hat{i} + 2\hat{j}$, so $(\nabla \times \vec{F}) \cdot \hat{n} = (2\hat{i} + 2\hat{j}) \cdot (-2\hat{i} - 2\hat{j} + \hat{k})/3 = -8/3$ and hence

$$\int_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS = -\frac{8}{3} \int_S 1 \, dS.$$ 

The region $S$ is an ellipse given by the graph of $z = 2x + 2y$ over the shadow region $R = \{(x+1)^2 + (y+1)^2 = 4\}$, so $dS = \sqrt{2^2 + 2^2 + 1^2} \, dA = 3 \, dA$ and hence \[\int_S 1 \, dS = 3 \cdot (\pi \cdot 4) = 12\pi,\] so

$$\int_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS = -32\pi.$$ 

Alternatively, parametrize $C$ as

$$\vec{r}(t) = (-1 + 2\cos(t))\hat{i} + (-1 + 2\sin(t))\hat{j} + (4\cos(t) + 4\sin(t) - 4)\hat{k}$$

and then one may compute that

$$\vec{F} \cdot d\vec{r} = -20\sin^2(t) + 10\sin(t) - 12\cos^2(t) + 6\cos(t)$$

and the integral from 0 to $2\pi$ of this is clearly $-20\pi + 0 - 12\pi + 0 = -32\pi.$

2. Instead we find the flux of $\nabla \times \vec{F} = 2(y+1)\hat{i} + 2(z+1)\hat{j} + 2(x+1)\hat{k}$ across the plane region $S$ with boundary $C$. There, $\hat{n} = -(\hat{i} + \hat{j} + \hat{k})/\sqrt{3}$ and $dS = \sqrt{3} \, dA$ so the flux becomes

$$\int_0^1 \int_0^{1-x} -2(x + y + z + 3) \, dy \, dx = \int_0^1 \int_0^{1-x} -2(1 + 3) \, dy \, dx = -4.$$ 

3. We have

$$\nabla \times \vec{F} = (x + 2)\hat{i} + (2y + 1)\hat{j} + (3 - 3z)\hat{k}$$

(a) We have

$$z = \frac{3}{2} \sqrt{4 - 4x^2 - y^2}$$

so

$$-f_x\hat{i} - f_y\hat{j} + \hat{k} = \frac{6x}{\sqrt{4 - 4x^2 - y^2}}\hat{i} + \frac{3y}{2\sqrt{4 - 4x^2 - y^2}}\hat{j} + \hat{k}$$

and hence

$$(\nabla \times \vec{F}) \cdot d\vec{S} = \left(\frac{6x^2 + 12x + 3y^2 + 3y/2}{\sqrt{4 - 4x^2 - y^2}} + 3 - \frac{9}{2} \sqrt{4 - 4x^2 - y^2}\right) \, dA$$

Upon performing the change of variables $x = \cos(t)$ and $y = 2r \sin(t)$ we see that $dA = 2r \, dr \, d\theta$ and the integrand becomes

$$(\nabla \times \vec{F}) \cdot d\vec{S} = \frac{6r^3 + 12r^2 \cos(t) + 6r^3 \sin^2(t) + 3r^2 \sin(t)}{\sqrt{1 - r^2}} + 6r - 18r \sqrt{1 - r^2} \, dr \, d\theta$$

and the flux is then

$$\int_0^1 \int_0^{\pi/2} \left(\frac{6r^3 + 12r^2 \cos(t) + 6r^3 \sin^2(t) + 3r^2 \sin(t)}{\sqrt{1 - r^2}} + 6r - 18r \sqrt{1 - r^2}\right) \, d\theta \, dr$$

$$= \int_0^1 (9/2)\pi r^3 + 15r^2 \sqrt{1 - r^2} + 3\pi r - 9\pi r \sqrt{1 - r^2} \, dr$$

$$= \frac{9\pi}{2} \left(\frac{2}{3}\right) + 15 \left(\frac{\pi}{4}\right) + \frac{3\pi}{2} - 9\pi \left(\frac{1}{3}\right)$$

$$= \frac{21\pi}{4}.$$
(b) For $S_{xy}$, we have $z = 0$ so $\mathbf{F} \cdot \hat{n} = \mathbf{F} \cdot \hat{k} = 3$ so the flux is thrice the area of this region, which is $3\pi(1)(2)/4 = 3\pi/2$. Similarly for $S_{yz}$ we get $2\pi(2)(3)/4 = 3\pi$ and for $S_{xz}$ we get $1\pi(1)(3)/4 = 3\pi/4$ so the total is $21\pi/4$.

(c) Let the boundary pieces be $C_{xy}$, $C_{yz}$, $C_{xz}$, where they are labeled according to which plane they belong.

Parametrize $C_{xy}$ by $\mathbf{r}(t) = \cos(t)\hat{i} + 2\sin(t)\hat{j}$ from $t = 0$ to $t = \pi/2$. Then $\mathbf{F}(t) = 3\cos(t)\hat{j} + (2\sin(t)\cos(t) - \cos(t))\hat{k}$, so that $\mathbf{F}(t) \cdot d\mathbf{r} = 6\cos^2(t)\, dt$. The integral from 0 to $\pi/2$ of this is $3\pi/2$.

Parametrize $C_{yz}$ by $\mathbf{r}(t) = 2\cos(t)\hat{j} + 3\sin(t)\hat{k}$ from $t = 0$ to $t = \pi/2$. Then $\mathbf{F}(t) = 18\sin(t)\cos(t)\hat{i} + (-6\sin(t))\hat{j}$, so that $\mathbf{F}(t) \cdot d\mathbf{r} = 12\sin^2(t)\, dt$. The integral from 0 to $\pi/2$ of this is $3\pi$.

Parametrize $C_{xz}$ by $\mathbf{r}(t) = \sin(t)\hat{i} + 3\cos(t)\hat{k}$ from $t = 0$ to $t = \pi/2$. Then $\mathbf{F}(t) = (3\sin(t) - 6\cos(t))\hat{j} - \sin(t)\hat{k}$, so that $\mathbf{F} \cdot d\mathbf{r} = 3\sin^2(t)\, dt$. The integral of this from 0 to $\pi/2$ is $3\pi/4$.

The sum of all these works is then $3\pi/2 + 3\pi + 3\pi/4 = 21\pi/4$. 