Today we discuss an application of CLI's developed in previous 4 lectures. We study uniformly random lozenge/domino tilings of polygonal domains. Handout shows examples. Phenomenology is similar and we stick to lozenge case.

Consider a polygonal domain on triangular grid (e.g. hexagon), fix mesh size $\frac{1}{L}$.

There are finitely many tilings with lozenges:

\[
\begin{array}{c}
\Diamond \\
\n\end{array}
\]

(Consider uniformly random tiling.)

**Question:** How does it look like as $L \to \infty$?

Answer is given in terms of **height function**

\[
H(x,y) = \text{number of paths below } (x,y).
\]

Draw paths:

\[
\begin{array}{c}
\n\end{array}
\]

(up to affine transforms) $\Rightarrow$ number of $\Diamond$ above $(x,y)$ (or below)
Theorem (Kohn- Kenyon- Propp - 2000)
For any polygon \( \lim_{L \to \infty} \frac{H_L(x,y)}{L} = h(x,y) \)
deterministic

\( h(x,y) \) is found as a unique maximizer of
\[ \int_{\text{poly}} \nabla h \, dA \]
with appropriate boundary conditions.

Theorem (Kenyon- Okounkov - 2006), Introduce complex slope
\[ z(x,y) \]

\[ e^{2 \pi i} \gamma \]

\[ p(0), p(\Omega), p(\Omega) \]

local proportions of 3 types of lozenges
partial derivatives of \( h \)
in appropriate directions

In "liquid region" all types of lozenges are present
and \( z \in \mathbb{C} \), i.e. it is non-real.

"Frozen" regions \( \rightarrow \) real \( z \). "Arctic curve" \( \rightarrow \) real
frozen/liquid border

Then \( z(x,y) \) is an algebraic function of \( (x,y) \).
It is found algorithmically
(Proven for simply-connected polygons. Believed to be true
in general.)
Conjecture (Kenyon-Okonkov)
In the liquid region \( \lim_{L \to \infty} H_2(a, b) - \mathbb{E} H_2(a, b) = G_2(x, y) \)

no scaling?

\[ \text{The Gaussian Free Field in } L \text{ with Dirichlet boundary conditions and in complex structure given by } z. \]

[No fluctuations in frozen regions]

[Perestrello"

Duits, Butkov-Gorin, Borodin-Ferrari], but in the most general form it is still open.

Theorem The conjecture is true for trapezoids and gluings of trapezoids along a single axis (combination of Butkov-Gorin, Borodin-Gorin-Guionnet).

Examples: Trapezoid

Gluing of 2 trapezoids

Hexagon (simply connected)

Holey Hexagon (topology of annulus)

Remark: height is fixed along the boundary of the hole)

What is GFF?

A) \( \text{Im } U = \{ \text{Im } A > 0 \} \) with standard complex structure

\( G(U) \) - generalized centered Gaussian function
\[ E \cdot G(N) \cdot G(N) = -\frac{1}{2\pi} \ln \left| \frac{N_1 - N_2}{N_1 - N_2} \right| \]

Green function of Laplace in \( U \).

The values of \( G \) are not defined, however for smooth \( f \) are bona fide centered gaussians.

\[ \int_{\partial D} f(z) \cdot G(z) \, d\gamma \]

\[ \int_{\Delta D} f(z) \cdot G(z) \, d\gamma \]

B) Simply-connected domain \( D \) with a complex structure \( z(x,y) \) (i.e. continuous complex function).

Consider \( \mathcal{N} : D \to \mathbb{U} \rightarrow \mathcal{U} \rightarrow \mathcal{U} \)

\[ (x,y) \rightarrow \mathcal{N}(x,y) \]

which is conformal w.r.t. \( z(x,y) \) [a bit more delicate]

\[ \text{when } \mathcal{N} \text{ is locally constant} \]

Riemann uniformization theorem: \( \mathcal{N} \) exists and is unique up to Möbius transformations.

Set \( G_z(x,y) = G(\mathcal{N}(x,y)) \)

"Pullback of GFF in \( \mathbb{U} \) by map \( \mathcal{N} \)"

C) A domain of more complicated topology: either find a green function of Laplace operator defined in a local coordinate system of complex slope \( \Delta D \) or map to a subset of plane (conformally) and take a pullback of GFF (= Gaussian with Green function) in this domain.
2) In what topology do we prove convergence? Values at points make no sense (need different rescaling)

We consider pointings with test functions

\[ f(y) \]

\[ \text{polyomorph} \]

\[ \int (H_L(x_0,y) - \mathcal{H}_L(x_0,y)) f(y) \ dy \rightarrow \]

\[ \int G(\mathcal{H}(x_0,y)) f(y) \ dy \]

Jointly in finitely many \( x_0, f \).

Such test functions are dense \( \Rightarrow \) uniquely define limit.

We will do two things:
1) Sketch heuristic argument for general domain
2) Sketch our rigorous argument.

Heuristics: How is LLN (variational principle) developed?

Take a height function \( \hat{h}(x,y) \)

\[ \mathbb{P} \left( \frac{H_L(x,y)}{L} \in \varepsilon \text{-neighborhood of } \hat{h}(x,y) \right) \approx \exp \left( \frac{-C L^2 \int S(\hat{h}) \ dy}{\varepsilon^2} \right) \]

where \( S(\hat{h}) = \frac{1}{3} (L(\hat{h}p(\hat{h}) + \hat{h}p(\hat{h}) + \hat{h}p(\hat{h})) \]

\[ L(t) = -\frac{3}{2} \log |2s_{x+t}^2| + \text{d}t \]

This has a proper formal meaning for LLN, but not for CLT.
If we simply consider a measure on height functions

\[ p(h) = \exp\left( L^2 SS \sigma(h) \right) \]

Then maximum of \( SS \) will give a typical height function. Linearizing near Taylor expand the function near extremum. Linear part will vanish (Euler-Lagrange). Quadratic will give \( \exp(\text{SS} G_2(h, \nabla h)) \).

In appropriate coordinate system this is \( \exp(-SS \langle \nabla (h - \hat{h}), \nabla (h - \hat{h}) \rangle) \) \( \langle h - \hat{h}, \nabla (h - \hat{h}) \rangle \).

If we see quadratic structure (Gaussians). Covariance \( \rightarrow \) inverse matrix/ operator \( \Rightarrow \) Green function (General formula for Gaussians).

\[ \text{Never made rigorous} \]

The actual proof.

Trapezoid

\[ \text{N lozenges on right border.} \]

The measure is uniform, conditionally on.
**Proposition:** The S.G.F. of $t$ lozenges at the section $x = t$ is

$$S_{x} (x_1, \ldots, x_t, 1^{N-t})$$

# of tilings of a trapezoid

$$S_{x} (1^{N})$$

**Proof:** This is the branching rule for Schur polynomials (i.e. what happens when we plug $x_N = 1$).

**Corollary:** We can apply the main theorem on S.G.F. to get gaussianity + covariance along a single section. Slight extension $\rightarrow$ joint limit.

Remains to identify double contour integral in the answer of [But-Gor] with double integral of GFF. (takes efforts, but doable)

**Holey hexagon**

Condition on horizontal lozenges along the axis.

$$P (e_1, \ldots, e_N) \sim$$

$$\sim S_{x \in \mathbb{F}} (1^{N}) S_{\mathbb{F}} (1^{B+C-D})$$

deterministic procedure applied to $\mathbb{F}$
Use 
\[ S_N(t_1, \ldots, t) = \lim_{q \to 1} S_n(t, q, \ldots, q^{N-1}) = \lim_{q \to 1} \frac{\eta(q \cdot t_{i+1} - \eta(q \cdot t_i))}{\log q} \]

Thus \( P(t_1, \ldots, t_N) \) is \( \beta = 2 \log y \). 

Virtual particles in \( \lambda \), \( \Sigma \) will create potential. 

Consider \( S.G.F. \) \( \sum P(t') \frac{S_N(t_1, \ldots, t_N)}{S_N(t_1, \ldots, t_N)} \frac{S_N(t_1, \ldots, t_N)}{S_N(t_1, \ldots, t_N)} \frac{S_N(t_1, \ldots, t_N)}{S_N(t_1, \ldots, t_N)} \)

Multivariate 
\[ S_p(x_1, x_2, y_1, y_2) \]

Slight extension of main theorem of [But-Gor] \( \Rightarrow \)

Partial derivatives of this \( S.G.F. \)

An analogue of proposition (x)

Joint \( S.G.F. \) along two such sections \( \beta + c - d - p \)

So we know it 0.

Again by a slight extension of the main theorem, we get CLT.

and covariance everywhere. Remains to identify.
And we get a pullback of GFF in this domain with annulus topology.