Today we sketch a proof of the theorem.

\[ P_N(\lambda_1, \ldots, \lambda_N) \rightarrow S_p = \sum \frac{P(N)}{S_N} \frac{S_N(x_1, \ldots, x_N)}{S_N(1, \ldots, 1)} \]

\[ I) \quad \frac{1}{N} \sum_{i=1}^{N} \left( \frac{\lambda_i + N - i}{N} \right)^k \rightarrow \rho(k) \]

\[ II) \quad \text{Covariance} \left( \sum \left( \frac{\lambda_i + N - i}{N} \right)^\nu, \sum \left( \frac{\lambda_i + N - i}{N} \right)^\eta \right) \rightarrow \text{cov}(\nu, \eta) \]

\[ III) \quad \text{Higher moments} \quad \sum \left[ \left( \frac{\lambda_i + N - i}{N} \right)^\nu - \mathbf{E} \left( \frac{\lambda_i + N - i}{N} \right) \right] \rightarrow \text{Wick's formula} \]

If and only if (Theorem ?)

\[ 1) \quad \frac{1}{N} \left( \frac{\partial}{\partial x_1} \right)^\nu \ln S_{pN} \mid_{x_1 = \ldots = x_N} \rightarrow C_k \quad \text{(What ?)} \]

\[ 2) \quad \left( \frac{\partial}{\partial x_1} \right)^\nu \left( \frac{\partial}{\partial x_2} \right)^\nu \ln S_{pN} \mid_{x_1 = \ldots = x_N} \rightarrow d_k, \nu \]

\[ 3) \quad \prod_{i=1}^{K} \left( \frac{\partial}{\partial x_i} \right)^\nu \ln S_{pN} \mid_{x_1 = \ldots = x_N} \rightarrow 0 \quad \text{if} \ |i| \geq 2 \]

**Formulas linking two sets of numbers**

How to extract moments from generating function?

\[ N = 1 \quad \sum_{k} x^k P(k) \leftarrow \mathbf{P}(3=k) = \mathbf{F}(x) \]

\[ (x \frac{\partial}{\partial x})^m \mathbf{F}(x) = \sum \nu^m x^k P(k) \]

Plug in \( x = 1 \) to get \( \mathbb{E} Z^m \).
What was the key ingredient?
A dif. operator $x \frac{\partial}{\partial x}$, whose eigenfunction is $x^k$.

We mimic the same for Schur gen. func.

$$\mathcal{D}_k = V(x)^{-1} \sum_{i=1}^N (x_i \frac{\partial}{\partial x_i})^k V(x), \text{ where}$$

$$V(x) = \prod_{i < j} (x_i - x_j). \text{ I.e. multiply, then differentiate, then divide.}$$

**Lemma.**

$$\mathcal{D}_k S_x = \sum_{i=1}^N (x_i + N-i)^k S_x$$

**Proof.**

$$S_x = \frac{\det (x_i^{x_j + N-i})_{i,j=1}^N}{V(x)}$$

$$V(x) S_x = \det (x_i^{x_j + N-i})_{i,j=1}^N$$

$\det =$ sum of monomials. Each monomial is an eigenfunction. When $\sum (x_i \frac{\partial}{\partial x_i})^k$ acts,

$\sum (x_i + N-i)^k$ appears.

**Corollary.**

$$\mathbb{E} \left( \sum_{i=1}^N (x_i + N-i)^k \right)^m = \left( \mathcal{D}_k \right)^m S_{P_n}(x_1, \ldots, x_n) \bigg|_{x_1=\ldots=x_n=1}$$

**Proof.** Same as $N=1$ case.
That's our way to compute moments. In the end, Theorem is based on this lemma. However, there is an important feature which makes theorems hard:

\( V(x) \) vanishes at \( x_s = \ldots = x_n = 1 \). We need to resolve the singularity.

We will prove only one step: \( 1), 2), 3) \Rightarrow N^{-k-1} \int_{-1}^{1} \sum (x_i + N^{-i})^n \rightarrow p(x) \)

**Proof.**

\[ \partial_{x_s} S_p |_{x_s = \ldots = x_n = 1} \]

How to apply \( \Pi (x_i - x_s)^{-1} \sum (x_i, 0_i)^x \Pi (x_i - x_j) \) to a function \( \partial_i (S_p) \)?

Each \( \partial_i \) acts on:

1) \( (x_i - x_s) \) turning it into 1.
2) On \( x_i \) from other (previous) \( x_i, 0_i \)
3) On \( S_p \)
4) On the result of previous differentiation of \( S_p \).

Write \( S_p = \exp (\ln S_p) \)

Then \( \partial_i S_p = \partial_i (\ln S_p) \cdot S_p \)
Conclusion

\[ \Delta u \cdot S_p \] is the sum of the terms of the form

\[ \sum_{i=1}^{x_n} \prod_{a=1}^{\infty} \frac{D_a (\ln S_p)}{(x_i - x_{i+1}) \cdots (x_i - x_{i+m})} \]

\[ (**) \]

Where \( q + m + \sum_{a=1}^{\infty} j_a = n. \)

(Is it clear?)

At this point we need to plug in \( x_1 = \cdots = x_n = 1. \)

\( S_p \) disappears. But the rest explodes.

(why?)

Simplest case: \[ \Delta u \bigg|_{x_n=2} \frac{x^2 \cdot D_1 (\ln S_p)}{x_1 - x_2} \]

\( (k=2, q=0, m=1) \)

Important: \( S_p \) and \( \Delta u \) are both symmetric?

Therefore, we also have the term \[ \frac{x^2 \cdot D_2 (\ln S_p)}{x_2 - x_1} \]

They sum up to

\[ \sum_{i=1}^{x_n} \prod_{a=1}^{\infty} \frac{D_a (\ln S_p)}{(x_i - x_{i+1}) \cdots (x_i - x_{i+m})} \]

\[ \frac{x^2 \cdot D_1 (\ln S_p)}{x_1 - x_2} \]

This has a well-defined limit \( x_1, x_2 \to 1 \) \( (\frac{0}{0}) \)

Which is \( (\text{Expressed through partial derivatives of } \ln S_p \text{ at } 1) \)

This extends to the general case!
Lemma: \( \text{Sym} (\ast) \) has a well-defined limit \( \sum_{j,j_2,...,j_m} \) at \( x_3 = ... = x_m = 1 \), which is a finite sum of products of partial derivatives of \( \ln S_p \) at 1.

Proof: Expand \( \ln S_p = \sum_{\rho} A^\rho (x_1^{-1})^{r_1_1} (x_2^{-1})^{r_2} ... (x_n^{-1})^{r_n} \) to reduce to polynomials.

For a polynomial we have:

\[
\text{Sym}_{x_3,..,x_n} \frac{f(x_3,..,x_n)}{(x_1-x_3) ... (x_1-x_n)} = \frac{1}{\prod_{i<j} (x_i-x_j)} \sum_{\sigma \in S_n} (-1)^{\sigma} G \left( \frac{f(x_1,..,x_n)}{(x_1-x_2) ... (x_1-x_n)} \prod_{i<j} (x_i-x_j) \right)
\]

This is a polynomial symmetric polynomial.

Hence, the ratio is a polynomial and therefore has a limit as \( x_3,..,x_n \to 1 \).

(Break?)

The terms \( \text{Sym} (\ast) \) make sense. Which of them give leading contribution?

1) For each "type" of term \( (\ast) \) there are \( \sim N^{m+1} \) such terms.

Why? # of ways to choose indices out of \( N \)
2) Each term gives contribution \( \leq N^\# \) non-zero \( z_a \).

Why? Because we want to have several factors with derivative w.r.t. 1 variable only. The factors are created by \( \partial_S \ln S_p \). Then we only differentiate, which does not create new factors.

So we have a maximization problem:

\[
q + m + \sum z_a = k. \quad m + 1 + \# z_a \rightarrow \max
\]

What's the solution?

\( q = 0, \quad z_1 = \ldots = z_{e} = 1, \quad \text{i.e. the term} \)

\[
x_1^k \left( \partial_x (\ln S_p) \right) \left[ \frac{1}{(x_1 - x_2) \ldots (x_1 - x_{m+1})} \right]
\]

Lemma: \( \lim_{x_i \to 1} \left( \frac{g(x_i)}{(x_1 - x_2) \ldots (x_1 - x_n)} \right) = 0 \)

Proof:

1) \( k < n-1 \) → gives 0 by degree consideration (we know, this is a polynomial 0.)

2) \( k > n-1 \) → gives 0 by comparing the multiplicities of 0 at \( x_i - 1 \)

3) \( k = n-1 \) \( \left( \frac{(y_i - x_i)^{n-1}}{(y_i - x_1) \ldots (y_i - x_{i-1})} + (n-1 \text{ term } x_i \cdots \cdots x_n) \right) \) is a constant! \( \chi_i \to \infty \Rightarrow \text{this constant is 1.} \)
\[ p(k) = \sum_{\varepsilon=0}^{\kappa} \left. \frac{u^1}{\varepsilon!} (p_{\varepsilon+1})! (k-\varepsilon)! \left( \frac{\partial}{\partial x} \right)^\varepsilon \left( (x+1)^k \phi(x)^{k-\varepsilon} \right) \right|_{x=0} = \]

\[ = \frac{1}{2 \pi i} \oint_0^{\infty} \frac{u^1}{\varepsilon!} \sum_{\varepsilon=0}^{\kappa} \frac{1}{\varepsilon!} \left( \frac{2+1}{\varepsilon+1} \right)^\varepsilon F(2)^{k-\varepsilon} \, dz = \]

\[ = \frac{1}{2 \pi i} \oint_0^{\infty} \frac{(2+1)^k F(2)^{k+1}}{\varepsilon+1} \sum_{\varepsilon=-1}^{\kappa} \left( \frac{\varepsilon+1}{\varepsilon+1} \right) \frac{1}{F(2)^{\varepsilon+1} z^{\varepsilon+1}} \, dz = \]

\[ e=-1, \text{no residue} \]

\[ = \frac{1}{2 \pi i} \oint_0^{\infty} \frac{(2+1)^k F(2)^{k+1}}{\varepsilon+1} \left( 1 + \frac{1}{F(2) z} \right)^{k+1} \, dz = \]

\[ = \left[ 2^{-1} \right] \frac{1}{k+1} \frac{1}{2+1} \left( F(2) \left( 2+1 \right) + \left( \frac{2+1}{2} \right)^{k+1} \left( 2+1 \right) \right) \]

And that's the expression for \( p(k) \) we had last time!