These lectures are about random $N$-particle discrete configurations $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{Z}$, as $N \to \infty$.

They come from several sources:

1) 2d statistical mechanics
2) Interacting particle systems
3) Asymptotic Representation Theory

1) → Handout  2) TASEP and relatives (one of them will be our running example)
3) Decomposition of reps of large groups into irreducible components (e.g., tensor products)

Although the methods will be quite general, but let us fix a running example to keep the discussion precise.

$N$ independent random walks with arbitrary initial conditions, jump probability $p$ and conditioned to have no collisions

Discrete analogue of $B=2$ Dyson Brownian Motion

\[ N = 3. \]
What's new in these lectures?

We develop robust methods, which lead to universal results (e.g., you do not care much about the initial condition in non-intersecting paths). This is different from the previous results of integrable probability, which were destroyed by small perturbations.

Our focus: macroscopic behavior / linear statistics

\[ \sum_{i=1}^{N} f \left( \frac{\lambda_i}{N} \right) \xrightarrow{N \to \infty} ? \]

(f - indicator \to particle counts, main developments)

LLN: \[ \frac{1}{N} \Theta (\cdot) \Rightarrow \text{const.} \]

CLT: Gaussian fluctuations

Two approaches \to two characterizations of Gaussian law.

\[ Z \sim N(0,1) \]

Characterizing equation

For smooth f

\[ \mathbb{E} f'(z) = \mathbb{E} z f'(z) \]

Integration by parts

Moments

\[ \mathbb{E} Z^n = \begin{cases} 0, & n \text{ is odd} \\ (n-1)!!, & n \text{ is even} \end{cases} \]

\# of perfect matchings of \{1,\ldots,n\}

We will develop "approximate exact formulas" through Schur generating functions

Discrete loop (Nekrasov - Borodin-Gorin-Gurvitch)

First 2 lectures
Setup for Nekrasov equations (simplest case).

\[ \lambda_n \in \mathbb{C}, \quad \Theta \geq 0 \]
\[ \ell_i = \lambda_i - \Theta i \quad a \geq \ell_1 > \ldots > \ell_N \geq b \]
\[ a - e_i \in \mathbb{Z}_{\geq 0}, \quad b - e_i \in \mathbb{Z}_{\geq 0} \]

\[ P(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z} \prod_{i<j} \left( \ell_i - \ell_j \right) \frac{\Gamma(\ell_i - \ell_j + \Theta)}{\Gamma(\ell_i - \ell_j + 1 - \Theta)} \prod_{i=1}^{N} w(\ell_i) \]

\( W \) - analytic, \( Z > 0 \) on \( (a, b) \), \( w(a) = w(b) = 0 \).

**Examples:**

\[ \Theta = 1 \quad \mapsto \prod_{i<j} (\ell_i - \ell_j)^2 \]

\( \ell_i \) - integers

**Initially packed with density \( \rho \) at time \( t \):**

\[ \frac{1}{Z} \prod_{i<j} (\ell_i - \ell_j)^2 \prod_{i=1}^{N} \left( \frac{N+1}{\ell_i - \Delta} \right)^{l_i-1} \frac{\Gamma(n+t)}{\Gamma(n+t-\ell_i+1)} \]

\( \Delta = 0, \quad b = N + t + 1 \).

For What? Kerlin-Negraro?

I guessed Viennot's formula for number of non-intersecting paths (interest?)

Other \( \Theta = 1 \) examples -> see handout.
1) Fix $\Theta, \ell, \ell_i \to \infty \quad (\ell_i - \ell_i) \quad \frac{1/(\ell_i - \ell_i + \Theta)}{1/(\ell_i - \ell_i + (1-\Theta))} \approx \approx (\ell_i - \ell_i) \quad 2\Theta

This is a discrete analogue of general $\beta$

log-gases in random matrix theory ($\beta=2\Theta$ = dimension of basis field).

2) Same interaction factor shows up in evaluation formulas for such symmetric functions (Rep. theory + integrable systems, Zonal polynomials, Calogero-Sutherland).

Important: $\ell_i = \lambda_i - \Theta i$ - very special "lattice"

$\ell_{i+1} - \ell_i \in \{\Theta, \Theta + 1, \Theta + 2, \ldots\}$

Main technical tool: "Nekrasov equation".

Assume $\frac{w(x)}{w(x-1)} = \frac{y^+(x)}{y^+(a)}$ = analytic in a neighborhood $[a, b]$.

$[w(a)=w(b)=0 \quad \Rightarrow \quad y^+(a) \neq y^+(a+1) = y^+(b) = 0]$.

Define $R_N(\ell) = y^-(\ell) \prod_{i=1}^{N} (1-\frac{\Theta}{3-\ell_i}) + \quad y^+(\ell) \prod_{i=1}^{N} (1+\frac{\Theta}{3-\ell_{i-1}})$.

Then $R_N(\ell)$ is analytic in $\mathbb{N}$, i.e. it has no poles.

This is an equation.
Guessing right form is hard originates in similar expressions in the work of Nekrasov and collaborators. Uses all features of the definition.

**Proof.** We compute the residue at a possible pole \( x = n - m \cdot \Theta \).

Fix \( i \in \{1, \ldots, N\} \) \( \bar{E} = \) sum over configurations those which have \( l_i = x \) and \( l_i = x-1 \) contribute to the pole 1st term 2nd term \( \bar{E}_i = x \) \( \bar{E}_i = x-1 \)

Take a configuration \( \bar{E} \) and let \( \bar{E}^- \) be obtained by \( l_i \to l_i - 1 \). Then

\[
\frac{P(\bar{E}^-)}{P(\bar{E})} = \frac{w(x-1)}{w(x)} \prod_{j \neq i} \frac{l_i - 1 - l_j}{l_i - l_j + \Theta - 1} = \frac{\varphi^-(x)}{\varphi^+(x)}
\]

Two cases \( j \leq i, j > i \) lead to same formula?

\[
\Theta \cdot P(\bar{E}^-) \varphi^+(x) \cdot \prod_{j \neq i} \left( 1 + \frac{\Theta}{l_i - l_j - 1} \right)
\]

= \( \Theta P(\bar{E}) \) \( \varphi^-(x) \cdot \prod_{j \neq i} \left( 1 - \frac{\Theta}{l_i - l_j} \right) \)

So residue at \( x \) from \( \bar{E} \) + from \( \bar{E}^- \) = 0
Important: $\bar{E}$ might fail to be in our state space $(\tilde{E}^y_{i-1})$ if $\bar{E}_i - E_{i-1} = 0$.

However, in this case $P(\bar{E}_i) = 0$ and r.h.s. $= 0$

$\Psi(x) = 0$ or $(1 - \frac{1}{E_i}) = 0$

Similarly, if $\bar{E}_i$ is in the state space but $\bar{E}$ is not. That's where all the features of the definition of $\tilde{E}_i$, discrete log-gass, play a role.

How to use it? First application: explicit LLN

Assume $w(x) = \exp(-N\frac{V(x)}{N} + \varepsilon_n)$, where

$V$ is continuous in $[\frac{a}{N}, \frac{b}{N}]$, and $|V''(x)| \leq C (1 + \ln|(3/4 - \frac{a}{N})| + \ln|(3/4 - \frac{b}{N})|)$

and $|\varepsilon_n| \leq C \ln(N).

Theorem: Then $w_i := \frac{1}{N} \sum E_i/n \approx m(x)dx$,

where $m(x)dx$ is the maximizer of $\int \ln(x-y) \mu(x) \nu(y) d\mu d\nu - \int V(x) V(x)dx$

$I[V] = \Theta SS \ln(x-y) \nu(x) \mu(y) d\mu d\nu - \int V(x) V(x) dx$

on probability measures on $[a/N, b/N]$, $0 \leq \mu(x) = e^{-\frac{x}{\theta}}$

(That's because $E_{i+1} - E_i \leq \Theta$)

in the form $\forall \varepsilon > 0$, Lipschitz in a neighborhood of $[a/N, b/N]$ function $f$

$N^{-\frac{1}{2} - \varepsilon} \left| \int f(x) g_{\bar{E}_i} (dx) - \int f(x) \mu(x) dx \right| \rightarrow 0$

in probability and in the sense of moments.
The proof is standard, as in continuous case, I will not present it, unless the audience is really interested. (Nothing to do with Nekrasov equations)

Given the theorem, what happens with Nekrasov equation?

Assumptions: \( z = N^2 \)

\[
\begin{align*}
\varphi^-_N(N^2) & \to \varphi^-(z) \\
\varphi^+_N(N^2) & \to \varphi^+(z)
\end{align*}
\]

Proposition:

\[
R_N(N^2) = \varphi^-_N(N^2) \prod (1 - \frac{\theta}{N^2-z_i}) + \varphi^+_N(N^2) \prod (1 + \frac{\theta}{N^2-z_i})
\]

\[
\Rightarrow \quad R(z) = \varphi^-(z) \exp(-\theta G(z)) + \varphi^+(z) \exp(\theta G(z))
\]

where \( G(z) = \int \frac{m(x)}{z-x} \, dx \) \( R(z) \) is analytic where \( \varphi^\pm \) are

Proof:

\[
\prod (1 - \frac{\theta}{N^2-z_i}) = \exp \left( \sum_{i=1}^{N} \ln \left( 1 - \frac{\theta}{N \left( z - z_i/N \right)} \right) \right) = \exp \left( -\theta \sum_{i=1}^{N} \frac{1}{z - z_i/N} + \frac{1}{N^2} \sum_{i=1}^{N} \frac{1}{(2 - z_i/N)^2} \cdot O(1) \right) \to
\]

for \( z \) bounded away from \([\frac{a}{N}, \frac{b}{N}]\)

by Theorem \( \exp(-\theta G(z)) \). Similarly, for the second integral expectation. So choose a contour \( \gamma \), such that \([\frac{a}{N}, \frac{b}{N}]\) is inside at positive distance.

\[
R_N(N^2) \to R(z) \text{ uniformly on } \gamma.
\]
For any \( w \) inside \( \gamma \)

\[
R_N(Nw) = \frac{1}{2\pi i} \oint_{\gamma} \frac{R_n(Nz)}{z - w} \, dz
\]

Cauchy formula - follows from analyticity of \( R_n \).

\( N \to \infty \) leads to

\[
R(w) = \frac{1}{2\pi i} \oint_{\gamma} \frac{R(z)}{z - w} \, dz
\]

which shows that \( R(w) \) is analytic in \( w \) and provider analytic continuation to everywhere inside \( \gamma \).

**Example:** Non-intersecting walks

\[
w(t) = P^{Nt-\alpha} (1-p)^{N+t-\alpha} \left( \frac{x^{N+t-\alpha}}{x-1} \right)
\]

on \( \{1, \ldots, N+t\} \)

\[
\frac{w(x)}{w(x-1)} = \frac{p}{1-p} \frac{N+t-x}{x-1}
\]

\[
= \frac{p}{1-p} \frac{1 + \frac{t-x}{N}}{\frac{x}{N} - \frac{1}{N}} \to \frac{p}{(1-p) \frac{1}{N}} = \psi^+(z)
\]

\[
\frac{1}{N} \to \frac{X}{N} \to \frac{X}{N} \to \frac{p(1+z)}{(1-p)z} = \psi^-(z)
\]

Therefore, the empirical measure of walks at time \( t \) as \( N \to \infty \) has law satisfying

\[
p(1+z) \exp \left( G(z) \right) + (1-p) \exp \left( -G(z) \right) = R(z) - \text{analytic in } z \in \mathbb{C}
\]

\[
z \to \infty \exp \left( G(z) \right) \approx \exp \left( \frac{1}{2} + o \left( \frac{1}{N} \right) \right) = \left( 1 + \frac{1}{2} + o \left( \frac{1}{N} \right) \right)
\]

Thus \( |R(z)| \leq C|z|, z \to \infty \).

Liouville theorem: then \( R(z) = az + b \).

On the other hand, \( R(z) = p(1+z) - (1-p)(1-\frac{1}{z}) + o \left( \frac{1}{z} \right) = \)

\[
= o \left( \frac{1}{z} \right)
\]

So \( \frac{1}{z} = \frac{1}{(1-2p)} \), \( b = p(1+c) - 1 \).
\[ p (1 + \varepsilon^2) \exp G(z) + (1-p)z \exp (-G(z)) = \]
\[ = (1-2p)z + p(1+\varepsilon^2) - 1. \]

\[ \hat{\nabla} \]
\[ (\exp(G(z)))^2 \quad p(1+\varepsilon^2) - (\exp(G(z))(1-2p)z + p(1+\varepsilon^2) - 1 + (1-p)z) = 0. \]

Solving quadratic equation, we get \( \exp(G(z)) \) and \( G(z) \) itself.

\[ \hat{\nabla} \]
\[ m(\chi) = \frac{1}{4\pi} \text{Im} \lim_{\varepsilon \to 0} G(x + i\varepsilon). \]

\[ \hat{\nabla} \]
\[ \hat{\theta} \\quad \text{Arg} \left( \exp(G(x + i\varepsilon)) \right) \]

What's next? We will prove CLT using the same Nekrasov equation.

Main difficulty: no control over analytic \( R_n(\hat{\theta}^3) \) \implies \text{need to get rid of it somehow.} \]