Abstract

For all $k \geq 1$, we show that the problem of deciding whether a graph is $k$-planar is NP-complete, extending the well-known fact that deciding 1-planarity is NP-complete.

A graph is said to be $k$-planar if it can be embedded in the plane such that each edge participates in at most $k$ crossings. The minimal $k$ for which $G$ is $k$-planar is called the local crossing number of $G$. In [3] and later independently in [4], it was shown that the problem of deciding whether a given graph is 1-planar is NP-complete. In this note, we use this result to show that testing $k$-planarity is NP-complete for all $k \geq 1$.

Theorem 1. Deciding whether a graph is $k$-planar is NP-complete.

Our proof of Theorem 1 proceeds in two simple steps: first we modify the construction of [3] to prove that testing $k$-planarity is hard in multigraphs, and then we reduce to the graph case via subdivision. We note that in [4] the authors sketch a way of modifying their reduction to give hardness of $k$-planarity testing in multigraphs, but this approach is quite involved, and filling in the details appears somewhat difficult. Here, we provide a one-page proof of this fact, building on the reduction in [3] instead.

Theorem 2. Deciding whether a multigraph is $k$-planar is NP-complete.

In order to motivate the proof of Theorem 2, we first give a brief review of the gadget used in [3] to prove hardness of testing 1-planarity via a reduction from 3-partition. The 3-partition decision problem asks whether or not a multiset of $3m$ integers can be partitioned into $m$ multisets such

\footnote{A third proof was later given in [5], based on the proof of NP-hardness of the crossing number problem in [1].}
that the sum of each is the same. Without loss of generality, one may assume that every integer is positive and strictly between a fourth and half of the desired sum (for details, see [2]). The reduction converts an instance $S = \{a_1, \ldots, a_{3m}\}, \sum_{i=1}^{3m} a_i = Bm, \frac{B}{4} < a_i < \frac{B}{2}, i = 1, \ldots, 3m$, of 3-partition into a graph $G_S = (V, E)$ defined as follows:

For each $a_i \in S$, create a star with $a_i + 1$ leaves. In addition, create two “double wheel” gadgets called the transmitter and collector (shown in Figure 1b), with $3m$ and $Bm$ vertices on each wheel, respectively. Each bold edge in the wheel represents a copy of $K_6$ (see Figure 1c). Add an edge between one leaf of each star and the transmitter center $t$, and between all other leaves and the collector center $c$. Finally, connect every third vertex of the outer wheel of the transmitter to every $B^{th}$ vertex of the outer wheel of the collector using a copy of $K_6$, resulting in a total of $m$ copies of $K_6$ connecting the transmitter and collector (see Figure 1a for an example).
The paths between the transmitter and collector consisting of copies of $K_6$ partition the plane into $m$ regions, which partition the stars. 1-planarity is possible if and only if in each region the integers corresponding to the stars in that region sum to $B$. The key fact used in the reduction is that any 1-planar embedding of $K_6$ is such that for all $u, v \in K_6$ there exists a $u - v$ path consisting exclusively of crossed edges, making copies of $K_6$ uncrossable. Refer to [3] for a rigorous proof of these facts. Let $kG$ be the multigraph obtained by duplicating every edge of $G$ $k$ times. We will first provide a short self-contained proof that $k K_6$ has a more general property for any $k \geq 1$, and then provide a short proof of Theorem 2. Our proof of Theorem 2 will restrict to cases in which $B \geq 5$ and $m \geq 3$, which is still an NP-hard set of 3-partition instances.

**Lemma 3.** Any $k$-planar embedding of $k K_6$ is such that for all $u, v \in k K_6$ there is a $u - v$ path consisting entirely of $k$-crossed edges.

**Proof.** Given a $k$-planar embedding of $k K_6$, suppose that two copies of $(u, v)$ partition the remaining four vertices into multiple regions. Then at least $3k$ edges cross these two copies of $(u, v)$, a contradiction. Therefore, no two copies of $(u, v)$ partition the other four vertices. Let $uvw$ be a cycle whose multi-edges separate the remaining three vertices, $x$ on one side and $y, z$ on the other. If such a cycle does not exist, then this would imply that $K_6$ is planar. The $k$ copies of $(x, y)$ must cross $k$ copies of an edge of $uvw$ (the same holds for $(x, z)$). Suppose w.l.o.g. that $(x, y), (u, v)$ and $(x, z), (u, w)$ cross. Then $(v, z)$ and $(w, y)$ must cross, as $(x, y), (u, v), (x, z), (u, w)$ are already $k$-crossed. Since $(x, y), (u, v), (x, z), (u, w), (v, z), (w, y)$ are all $k$-crossed, $k K_6$ has our desired property (see Figure 2). 

\[\bbox[white]{\square}\]
Proof of Theorem 2: Given a graph $G_S$, it suffices to prove that if $G_S$ is not 1-planar, then $kG_S$ is not $k$-planar (the converse is immediately true). Lemma 3 ensures that $kK_6$ cannot be crossed in $kG_S$, and therefore, the $m$ “paths” consisting of copies of $kK_6$ partition the sphere (and the set of stars) into $m$ different regions. The ordering of the paths must agree with the original labeling, otherwise an edge would cross one of these “paths”. In each region, there are four multi-edge edge-disjoint paths which separate $t$ and $c$, two of length three and two of length $B$ (because $m \geq 3$). If one region had more than three stars, there would be at least four multi-edge edge-disjoint paths between the $t$ and $c$ in that region, a contradiction.

Because $G_S$ is not 1-planar, there does not exist a partition of the stars into $m$ regions such that for each region the sum of the integers corresponding to the stars in that region is $B$. Consider one such region which sums to more than $B$. The two multi-edge paths of length three separate every star center from $t$ (because $B \geq 5$). Alternatively, the two multi-edge paths of length $B$ must separate every star center from $c$, otherwise there would be a path of length two from $t$ to a star center that would have to cross at least three separating multi-edge paths, a contradiction. However, there are more than $B$ multi-edge edge-disjoint paths from star centers to $c$, all of which must cross a multi-edge path of length $B$, a contradiction. The proof
Figure 4: Local crossing number is not multiplicative. Here, the local crossing number of $G$ is 2, while the local crossing number of $2G$ is 3.

is complete.

**Remark:** Given the proof of Theorem 2, it is reasonable to ask whether the local crossing number of $kG$ is $k$ times the local crossing number of $G$. This turns out to be false in general, as demonstrated by the graph in Figure 4.

The perceptive reader may note that we have not used the full machinery of the gadgets. This is because restrictions of $B \geq 5$ and $m \geq 3$ allow a simpler proof. We could have created much simpler gadgets, but thought it best to conform exactly to the structure in [3]. We are now prepared to provide a short proof of our main result.

**Proof of Theorem 1:** We reduce multigraph $2k$-planarity testing to graph $k$-planarity testing in a straightforward way: given an instance $G$ of multigraph $2k$-planarity testing, construct the graph $G_*$ by subdividing each edge of $G$ (i.e. replacing $e = (u, v)$ by $(u, x_e), (x_e, v)$, where $x_e$ is a unique vertex for each $e$). If $G_*$ is $k$-planar, then taking any $k$-planar embedding of $G_*$ and reversing the subdivision operation give an embedding of $G$ which is clearly $2k$-planar. Conversely, given a $2k$-planar embedding of $G$, we can obtain from it a $k$-planar embedding of $G_*$ by placing the $x_e$ vertices “in the middle” of the crossings on $e$ so that each segment has at most $k$ crossings, after possibly perturbing the drawing of $G$ slightly so that no three edges
cross at a single point.

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References


