# Poisson varieties and D-modules (1)

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# Outline (2)

I. "Classical" story: Quotient singularities; deformation and resolution

II. "Quantum" story: Noncommutative deformations; Poisson varieties; theorems on finite-dimensional irreducible representations

III. Poisson traces and applications

IV. $D$-module approach

# Part I: Classical story.
# Quotient singularities (3)

- Let $V$ be a complex vector space (or smooth variety or complex manifold).
- Let $G$ be a finite group acting on $V$.
- Then, one has the *quotient singularity* $X = V/G$.
- Question 1: To what extent can we recover $G$ and $V$ from $X$?
- Question 2: What do the algebraic invariants of $X$ mean?

# Example: Kleinian singularities (4)

Example: $X = \mathbf{C}^2/G$, where $G < \mathrm{SL}_2(\mathbf{C})$ (a *Kleinian singularity*).

- Case $G = \mathbf{Z}/2$ acting by $\pm\,\mathrm{Id}$: $X$ is a singular conic ($x^2 + y^2 = z^2$ inside $\mathbf{C}^3$). Alternatively:
  - $X =$ nilpotent traceless $2 \times 2$ matrices,
    $$\mathrm{Nil}(\mathfrak{sl}_2) = \{a^2 + bc = 0\} \subseteq \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbf{C} \right\} = \mathfrak{sl}_2.$$
- General case: there is a beautiful classification:

## Theorem (McKay correspondence I)

*The subgroups $G$ fall into three families (let $\zeta_n := e^{2\pi i/n}$):*

- *Type $A_{n-1}$: $G = \mathbf{Z}/n = \left\langle \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix} \right\rangle$; $X \subseteq \mathrm{Nil}(\mathfrak{sl}_n)$*
- *Type $D_{n+2}$: $G = \left\langle \mathbf{Z}/2n, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$; $X \subseteq \mathrm{Nil}(\mathfrak{so}_{2(n+2)})$*
- *Types $E_6, E_7, E_8$: preimage of tetrahedral, octahedral, and icosahedral rotation groups under $\mathrm{SU}_2(\mathbf{C}) = \mathrm{Spin}_3 \overset{2:1}{\twoheadrightarrow} \mathrm{SO}_3(\mathbf{R})$ ($X \subseteq \mathrm{Nil}(\mathfrak{e}_6), \mathrm{Nil}(\mathfrak{e}_7), \mathrm{Nil}(\mathfrak{e}_8)$)*

# Resolution of singularities (5)

- How to recover $G$ from $X = \mathbf{C}^n/G$?
- Try *resolution of singularities*: smooth $\widetilde{X}$ with $\pi : \widetilde{X} \twoheadrightarrow X$: surjective, injective over smooth part; compact fibers (proper).
- Example: $\pi : T^*\mathbf{C}P^1 \twoheadrightarrow \mathbf{C}^2/(\mathbf{Z}/2)$. $\pi^{-1}(0) \cong \mathbf{C}P^1$, $\dim H^2(T^*\mathbf{C}P^1) = 1$: the cycle around the cone becomes nontrivial.
- Great for $X = \mathbf{C}^2/G$: there is a unique minimal resolution $\widetilde{X}$, coming from "Springer resolution" of $\mathrm{Nil}(\mathfrak{g}) \supseteq X$; $\pi^{-1}(0) = \#(\mathrm{Irreps}(G)) - 1 = \dim H^2(\widetilde{X})$ copies of $\mathbf{C}P^1$; intersection graph is the Dynkin diagram of $\mathfrak{g} \supseteq X$.
- For $n > 2$, though, **a minimal $\widetilde{X}$ does not exist**. Partial fix: "crepant"; still doesn't exist in general (for $n > 3$, it is rare).

# Deformations (6)

- Alternative: Deform $X$, rather than resolving.
- Deformations of $X = \mathbf{C}^2/G$ are classified (difficult). Generic deformation is smooth, topologically $\cong \widetilde{X}$.
- But **(nice) smooth deformations don't exist in general** (for $G < \mathrm{Sp}(n)$, equivalent to existence of "crepant"; rare). Hard to classify.

**End of Part I.**

# Part II: "Quantum" story. Noncommutative deformations (7)

- Instead, consider *noncommutative* deformations:
- Let $X \subseteq \mathbf{C}^N$ (affine), $\mathcal{O}_X =$ global functions on $X$.
- Suppose $\mathcal{O}_X = \bigoplus_{m \geq 0} (\mathcal{O}_X)_m$ is graded ($X$ conical).
- Consider a *filtered* noncommutative deformation, i.e., $A = \bigcup_m A_{\leq m}$ an associative (noncommutative) algebra such that $\operatorname{gr} A := \bigoplus_m A_{\leq m}/A_{\leq (m-1)} \cong \mathcal{O}_X$.
- Then obtain $\{-,-\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$, from $\{\operatorname{gr}_m a, \operatorname{gr}_n b\} := \operatorname{gr}_{m+n-d}[a,b]$ (for all $a \in A_{\leq m}, b \in A_{\leq n}$; fixed $d \geq 1$).
- Straightforward: $\{-,-\}$ is a Lie bracket satisfying $\{fg, h\} = f\{g,h\} + g\{f,h\}$. Called *Poisson bracket*.

# Poisson varieties (8)

**Definition**

An affine Poisson variety is $X \subseteq \mathbf{C}^N$ with a Poisson bracket on $\mathcal{O}_X$.

**Definition**

A (filtered) quantization of a graded Poisson algebra $\mathcal{O}_X$ is a filtered $A$ such that $\operatorname{gr} A \cong \mathcal{O}_X$ as Poisson algebras.

**Example**

- $X = \mathbf{C}^{2n}$, $\mathcal{O}_X = \mathbf{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $\{f, g\} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}$.
- Quantization: Weyl algebra $\operatorname{Weyl}(\mathbf{C}^{2n})$ = algebra generated by $x_i, y_i$ with relations $[x_i, y_j] = \delta_{ij}$, $[x_i, x_j] = 0 = [y_i, y_j]$.

Note: $\operatorname{Weyl}(\mathbf{C}^{2n}) \cong$ ring of polynomial differential operators on $\mathbf{C}^n$.

**Example**

- If $G < \operatorname{Sp}(2n)$, then $X = \mathbf{C}^{2n}/G$, $\mathcal{O}_X := \mathcal{O}_{C^{2n}}^G$ is Poisson;
- Quantization: $\operatorname{Weyl}(\mathbf{C}^{2n})^G$. **Always exists, "smooth"!**

# Irreducible finite-dimensional representations (9)

### Example

For $X = \mathbf{C}^2/(\mathbf{Z}/2) = \mathrm{Nil}(\mathfrak{sl}_2)$, $\mathcal{O}_X = \mathrm{Sym}(\mathfrak{sl}_2)/\,\mathrm{Sym}(\mathfrak{sl}_2)_+^{\mathfrak{sl}_2}$.
Quantizations: $U(\mathfrak{sl}_2)/(C-\lambda)$, where $C = \text{Casimir} \in U(\mathfrak{sl}_2)^{\mathfrak{sl}_2}$.
Representations = reps of $\mathfrak{sl}_2$ with Casimir acting by $\lambda$.
At most one finite-dimensional irrep!

### Theorem (Follows from, e.g., Alev and Lambre)

*Let $A$ be an arbitrary quantization of $\mathcal{O}_X$, $X = \mathbf{C}^2/G$. Then $\#$f.d. irreps$(A) \leq \#$irreps$(G) - 1 = \dim H^2(\widetilde{X})$.*

Generally, a *Slodowy slice* $X_e \subseteq \mathrm{Nil}(\mathfrak{g})$ is a transverse slice to a coadjoint orbit at $G \cdot e$, Lie $G = \mathfrak{g}$. Includes $X$ as above and $\mathrm{Nil}(\mathfrak{g})$ itself. Resolved by Springer resolution $\pi : T^*\mathcal{B}_{\mathfrak{g}} \twoheadrightarrow \mathrm{Nil}(\mathfrak{g})$.

### Theorem (Dodd; Etingof-S. '10)

*Let $A$ be an arbitrary quantization of $X_e \subseteq \mathrm{Nil}(\mathfrak{g})$. Then $\#$f.d. irreps$(A) \leq \dim H^{\dim X_e}(\widetilde{X_e}) = \dim H^{\dim X_e}(\pi^{-1}(e))$.*

**End of Part II.**

# Part III: Poisson traces: proof of theorems

## Poisson and Hochschild traces (10)

Let $A$ be an associative algebra and $\mathcal{O}_X$ a Poisson algebra.

### Definition

$\mathsf{HH}_0(A) := A/[A,A]$ "zeroth Hochschild homology;"
$\mathsf{HH}_0(A)^* = \{\varphi : A \to \mathbf{C} \mid \varphi([a,b]) = 0, \forall a,b\}$ "Hochschild traces."

- If $\rho : A \to \mathrm{End}(V)$ is a f.d. rep, then $\mathrm{tr}(\rho) \in \mathsf{HH}_0(A)^*$.
- Standard theorem: If $\rho_1, \ldots, \rho_n$ distinct f.d. irreps, then $\mathrm{tr}(\rho_1), \ldots, \mathrm{tr}(\rho_n) \in \mathsf{HH}_0(A)^*$ are lin. independent.
- Corollary: $\#$f.d. irreps$(A) \leq \dim \mathsf{HH}_0(A)$.

### Definition

$\mathsf{HP}_0(\mathcal{O}_X) := \mathcal{O}_X/\{\mathcal{O}_X, \mathcal{O}_X\}$ "zeroth Poisson homology."
$\mathsf{HP}_0(\mathcal{O}_X)^* = \{\varphi : \mathcal{O}_X \to \mathbf{C} \mid \varphi|_{\{\mathcal{O}_X,\mathcal{O}_X\}} = 0\}$ "Poisson traces."

- Proposition: If $\mathrm{gr}\, A = \mathcal{O}_X$, then $\mathsf{HP}_0(\mathcal{O}_X) \twoheadrightarrow \mathrm{gr}\, \mathsf{HH}_0(A)$.
- Proof: $\mathrm{gr}[A,A] \supseteq \{\mathcal{O}_X, \mathcal{O}_X\}$.
- $\therefore \#$irreps$(A) \leq \dim \mathsf{HP}_0(\mathcal{O}_X)$; fixed bound for arbitrary $A$ !

# Proof of theorems (11)

Now the results follow immediately from:

**Theorem (Alev and Lambre '98)**

*Let* $X = \mathbf{C}^2/G$*. Then*
$\dim \mathrm{HP}_0(\mathcal{O}_X) = \#\mathit{irreps}(G) - 1 = \dim H^2(\widetilde{X})$.

Let $X_e \subseteq \mathrm{Nil}(\mathfrak{g})$ be a Slodowy slice and $\pi : T^*\mathcal{B}_\mathfrak{g} \twoheadrightarrow \mathrm{Nil}(\mathfrak{g})$ the Springer resolution. $\widetilde{X_e} := \pi^{-1}(X_e)$.

**Theorem (Etingof-S. '10)**

$\dim \mathrm{HP}_0(\mathcal{O}_{X_e}) = \dim H^{\dim X_e}(\widetilde{X_e}) = \dim H^{\dim X_e}(\pi^{-1}(e))$.

Different generalization of $X = \mathbf{C}^2/G$:
$S^n X = X^n/S_n = \mathbf{C}^{2n}/(G^n \rtimes S_n)$.

**Theorem (Etingof-S. '09)**

*As graded algebras in* $|t| = 1$,
$\bigoplus_{n\geq 0} t^n \mathrm{HP}_0(\mathcal{O}_{S^n X})^* \cong \mathrm{Sym}(\bigoplus_{m\geq 1} t^m \cdot \mathrm{HP}_0(\mathcal{O}_X)^*)$.

Corollary: $\#\text{irreps}(\text{quant. of } \mathcal{O}_{S^n X}) \leq \dim \mathrm{HP}_0(\mathcal{O}_{S^n X})$
$= \text{coeff. of } t^n \text{ in } \prod_{m\geq 1} \frac{1}{(1-t^m)^{\dim \widetilde{X}}} = \dim H^{2n}(\mathrm{Hilb}^n \widetilde{X} = \widetilde{S^n X})$.

**End of Part III.**

# Part IV: $D$-modules and the finiteness theorem (12)

What about general finite groups $G < \mathrm{Sp}(2n)$, $X = \mathbf{C}^{2n}/G$?

Theorem (Berest, Etingof, Ginzburg '06)

$\mathrm{HP}_0(\mathcal{O}_{\mathbf{C}^{2n}/G})$ *is finite-dimensional.*

Recall: A smooth Poisson variety $X$ is *symplectic* if the bracket $\{-,-\}$ induces a nondegenerate pairing $T_X^* \times T_X^* \to \mathcal{O}_X$ everywhere on $X$ ($T_X^* =$ cotangent bundle).
A (locally closed) *Poisson subvariety* $Y \subseteq X$ is a subvariety such that $\{-,-\}$ descends from $\mathcal{O}_X$ to $\mathcal{O}_Y$.
A Poisson variety $X$ has *finitely many symplectic leaves* if $X$ is a disjoint union of finitely many locally closed symplectic subvarieties. Each such subvariety is then a *symplectic leaf*.

Theorem (Etingof-S. '09)

*If $X$ has finitely many symplectic leaves, then* $\mathrm{HP}_0(\mathcal{O}_X)$ *is finite-dimensional.*

# D-modules (13)

Let $\xi_f$ be the vector field $\xi_f(g) = \{f, g\}$.
The Poisson traces $\varphi \in \mathsf{HP}_0(\mathcal{O}_X)^*)$ are the solutions in $\mathcal{O}_X^*$ to the equations $\xi_f(\varphi) = 0, \forall f \in \mathcal{O}_X$. Hence:
$\mathsf{HP}_0(\mathcal{O}_X)^* = \mathrm{Hom}_{D\text{-mod}}(M(X), \mathcal{O}_X^*)$,
where $M(X) =$ the quotient $D_X/(\xi_f \cdot D_X)_{f \in \mathcal{O}_X}$.
When $X$ is smooth, $D_X =$ ring of differential operators on $X$ and $(\xi_f \cdot D_X)_{f \in \mathcal{O}_X}$ is a right ideal.
There is a notion of *holonomic D-module*: a finitely-generated $D$-module on $X$ whose support in $T^*X = \mathrm{Spec}(\mathrm{gr}\, D_X)$ has dimension $\dim X$. Solutions to these are finite-dimensional.

## Theorem (Etingof-S. '09)

*If $X$ has finitely many symplectic leaves, then $M(X)$ is holonomic.*

Note: $\mathsf{HP}_0(\mathcal{O}_X) = M(X) \otimes_{D_X} O_X = H_0(p_*(M(X))) =$ underived pushfoward of $M(X)$ to a point, $p : X \to \{\mathrm{pt}\}$.
So, one can deduce either $\mathsf{HP}_0$ or $\mathsf{HP}_0^*$ f.d. $\Rightarrow$ finiteness thm.

# Proof of holonomicity theorem. (14)

Proof.

- At each $x \in X$, $(\xi_f)_x$ generate all directions tangent to the symplectic leaf through $x$.
- So the support of $M(X)$ in $T_x^*X \subseteq T^*X$ is perpendicular to the symplectic leaf through $x$.
- Thus, the support of $M(X)$ lies in the (finite) union of conormal bundles to the symplectic leaves, each of which has dimension $\dim X$. □

- Remark: $M(X)$ makes sense whether or not $X$ is affine. The holonomicity theorem applies generally, as does the proof.
- Also, the above shows that the composition factors of $M(X)$ are (IC) extensions of local systems on symplectic leaves ($M(X)$ is finite-length since it is holonomic).

# A conjecture on symplectic resolutions (15)

- Example: if $X$ is smooth and symplectic, $M(X) = \Omega_X$, the canonical bundle of volume forms.
- Why? Anything invariant under the flow of $\xi_f$ is constant ($\xi_f$ flow in all directions everywhere by nondegeneracy).

The following is proved in the case of Slodowy slices, and was used in proving the theorem above on its $\mathrm{HP}_0$.

## Conjecture (Etingof-S.)

*If $\pi : \widetilde{X} \to X$ is a symplectic resolution ($\mathcal{O}_X \to \mathcal{O}_{\widetilde{X}}$ is Poisson, a resolution, and $\widetilde{X}$ is symplectic), then $\pi_* \Omega_{\widetilde{X}} = M(X)$.*

The conjecture would imply: $\mathrm{HP}_0(\mathcal{O}_X) \cong H^{\dim X}(\widetilde{X})$.
More generally, it would imply
$H_i(p_*(M(X))) \cong H^{\dim X - i}(\widetilde{X})$ for all $0 \leq i \leq \dim X$.
Here $p : X \to \{\mathrm{pt}\}$ is the projection to a point.
Define: $\mathrm{HP}_i^{DR}(\mathcal{O}_X) := H_i(p_*(M(X)))$
"*Poisson-de Rham homology*" of $X$.
Algebraic invariant of $X$ capturing geometry of $\widetilde{X}$!
**End of talk.**