Poisson varieties and D-modules (1)

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Based on joint work with P. Etingof:
arXiv:1004.4634, arXiv:0908.3868, arXiv:0907.1715, and in progress

Outline (2)

I. "Classical" story: Quotient singularities; deformation and resolution

II. "Quantum" story: Noncommutative deformations; Poisson varieties; theorems on finite-dimensional irreducible representations

III. Poisson traces and applications

IV. *D*-module approach

Part I: Classical story.
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- Question 2: What do the algebraic invariants of X mean?

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- *Types E_6, E_7, E_8: preimage of tetrahedral, octahedral, and icosahedral rotation groups under $\mathrm{SU}_2(\mathbf{C}) = \mathrm{Spin}_3 \overset{2:1}{\twoheadrightarrow} \mathrm{SO}_3(\mathbf{R})$ ($X \subseteq \mathrm{Nil}(\mathfrak{e}_6), \mathrm{Nil}(\mathfrak{e}_7), \mathrm{Nil}(\mathfrak{e}_8)$)*

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- For $n > 2$, though, **a minimal $\widetilde{X}$ does not exist**. Partial fix: "crepant"; still doesn't exist in general (for $n > 3$, it is rare).

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- Deformations of $X = \mathbf{C}^2/G$ are classified (difficult). Generic deformation is smooth, topologically $\cong \widetilde{X}$.
- But **(nice) smooth deformations don't exist in general** (for $G < \mathrm{Sp}(n)$, equivalent to existence of "crepant"; rare). Hard to classify.

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- Consider a *filtered* noncommutative deformation, i.e., $A = \bigcup_m A_{\leq m}$ an associative (noncommutative) algebra such that $\operatorname{gr} A := \bigoplus_m A_{\leq m} / A_{\leq (m-1)} \cong \mathcal{O}_X$.

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- Then obtain $\{-,-\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$, from $\{\operatorname{gr}_m a, \operatorname{gr}_n b\} := \operatorname{gr}_{m+n-d}[a, b]$ (for all $a \in A_{\leq m}, b \in A_{\leq n}$; fixed $d \geq 1$).

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- Straightforward: $\{-,-\}$ is a Lie bracket satisfying $\{fg, h\} = f\{g,h\} + g\{f,h\}$. Called *Poisson bracket*.

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- Quantization: $\mathrm{Weyl}(\mathbf{C}^{2n})^G$. **Always exists, "smooth"!**

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Theorem (Dodd; Etingof-S. '10)

Let A be an arbitrary quantization of $X_e \subseteq \mathrm{Nil}(\mathfrak{g})$. Then $\#$f.d. irreps$(A) \leq \dim H^{\dim X_e}(\widetilde{X_e}) = \dim H^{\dim X_e}(\pi^{-1}(e))$.

End of Part II.

Poisson and Hochschild traces (10)

Let A be an associative algebra and $\mathcal{O}_X$ a Poisson algebra.

Definition

$\mathrm{HH}_0(A) := A/[A, A]$ "zeroth Hochschild homology;"
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Part III: Poisson traces: proof of theorems

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What about general finite groups $G < \mathrm{Sp}(2n)$, $X = \mathbf{C}^{2n}/G$?

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- Also, the above shows that the composition factors of $M(X)$ are (IC) extensions of local systems on symplectic leaves ($M(X)$ is finite-length since it is holonomic).

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Here $p : X \to \{\mathrm{pt}\}$ is the projection to a point.
Define: $\mathrm{HP}_i^{DR}(\mathcal{O}_X) := H_i(p_*(M(X)))$
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A conjecture on symplectic resolutions (15)

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The following is proved in the case of Slodowy slices, and was used in proving the theorem above on its HP_0.

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If $\pi : \widetilde{X} \to X$ is a symplectic resolution ($\mathcal{O}_X \to \mathcal{O}_{\widetilde{X}}$ is Poisson, a resolution, and $\widetilde{X}$ is symplectic), then $\pi_* \Omega_{\widetilde{X}} = M(X)$.

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End of talk.