Poisson traces on symmetric powers of symplectic varieties and type D_n Weyl group singularities (1)

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This talk is posted at `http://math.mit.edu/~trasched/`

Basic concepts (2)

- A Poisson algebra is a commutative algebra with a Lie bracket $\{-,-\}$ satisfying $\{fg,h\} = f\{g,h\} + g\{f,h\}$.
- An affine Poisson variety is an affine variety $\operatorname{Spec} \mathcal{O}_X$ where $\mathcal{O}_X$ is a Poisson algebra.
- Example: an affine symplectic variety (X,ω): X is smooth, ω is a closed nondegenerate two-form.
  - Nondegeneracy means: ω induces an isomorphism $\tilde{\omega} : T_X \to T_X^*$.
  - The Poisson bracket is obtained from the inverse of this isomorphism ($\{f,g\} = \tilde{\omega}^{-1}(df)(g)$).
  - Conversely, nondegenerate smooth Poisson = symplectic.
- A star-product quantization of $\mathcal{O}_X$ is an associative product $\star$ on $\mathcal{O}_X[\![\hbar]\!] = \{\sum_{m\geq 0} a_m \hbar^m : a_m \in \mathcal{O}_X\}$ satisfying:
  - $a \star b \equiv ab \pmod{\hbar}$;
  - $a \star b - b \star a \equiv \hbar\{a,b\} \pmod{\hbar^2}$.
- Similarly, if $\mathcal{O}_X$ is a graded Poisson algebra, we can consider a filtered quantization: $A = \bigcup_{m\geq 0} A_{\leq m}$ is a filtered algebra, $\operatorname{gr} A := \bigoplus_{m\geq 0} A_m/A_{m-1} \cong \mathcal{O}_X$ ($A_{-1} := 0$), and $\{\operatorname{gr}_m a, \operatorname{gr}_n b\} = \operatorname{gr}_{m+n-1}[a,b]$ for $a \in A_{\leq m}, b \in A_{\leq n}$.

Goals and motivation (3)

- Main goal: describe $\mathrm{HP}_0(\mathcal{O}_X) := \mathcal{O}_X/\{\mathcal{O}_X, \mathcal{O}_X\}$ when X is the affine Poisson variety $X = S^n Y = Y^n/S_n$, for Y affine symplectic (and higher versions $\mathrm{HP}_i^{DR}(X), i > 0$).
  - For X symplectic, $\mathrm{HP}_0(\mathcal{O}_X) \cong H^{\dim X}(X)$, and $\mathrm{HP}_*^{DR}(X) \cong H^{\dim X - *}(X)$.
- Consequences for *all* quantizations of these varieties: bounds on Hochschild homology ($\dim \mathrm{HH}_0$), and on numbers of finite-dimensional irreps and prime ideals.
- More generally, study the D-module on $X = S^n Y$ describing Hamiltonian flow invariance, whose solutions in $\mathcal{O}_X^*$ are $\mathrm{HP}_0(\mathcal{O}_X)^*$. (Hamiltonian flow = along vector fields $\{f, -\}$.)
  - Surface case: higher solutions $\mathrm{HP}_*^{DR}(X)^*$ are $H^{n-*}(\mathrm{Hilb}^n Y)^*$.
- Conjectures on symplectic resolutions
- Results on type D_n quotient singularities, i.e., $T^*\mathbf{C}^n/W$ with W a type D_n Weyl group acting on $\mathbf{C}^n$.

Poisson and Hochschild traces (4)

Let A be an associative algebra and $\mathcal{O}_X$ a Poisson algebra.

Definition

$\mathrm{HH}_0(A) := A/[A, A]$ "zeroth Hochschild homology;"
$\mathrm{HH}_0(A)^* = \{\varphi : A \to \mathbf{C} \mid \varphi([a, b]) = 0, \forall a, b\}$ "Hochschild traces."

- If $\rho : A \to \mathrm{End}(V)$ is a f.d. rep, then $\mathrm{tr}(\rho) \in \mathrm{HH}_0(A)^*$.
- Standard theorem: If $\rho_1, \ldots, \rho_n$ distinct f.d. irreps, then $\mathrm{tr}(\rho_1), \ldots, \mathrm{tr}(\rho_n) \in \mathrm{HH}_0(A)^*$ are lin. independent.
- Corollary: $\#$f.d. irreps$(A) \leq \dim \mathrm{HH}_0(A)$.

Definition

$\mathrm{HP}_0(\mathcal{O}_X) := \mathcal{O}_X/\{\mathcal{O}_X, \mathcal{O}_X\}$ "zeroth Poisson homology."
$\mathrm{HP}_0(\mathcal{O}_X)^* = \{\varphi : \mathcal{O}_X \to \mathbf{C} \mid \varphi|_{\{\mathcal{O}_X, \mathcal{O}_X\}} = 0\}$ "Poisson traces."

- Proposition: If $A_\hbar = (\mathcal{O}_X[\![\hbar]\!], \star)$ is a star-product quantization of $\mathcal{O}_X$, then $\mathrm{HP}_0(\mathcal{O}_X)(\!(\hbar)\!) \twoheadrightarrow \mathrm{gr}_\hbar \mathrm{HH}_0(A[\hbar^{-1}])$.
- Proof: $[A_\hbar, A_\hbar] + \hbar^2 A_\hbar \supseteq \hbar\{\mathcal{O}_X, \mathcal{O}_X\} + \hbar^2 A_\hbar$.

Consequences in noncommutative algebra (5)

- $\#$f.d. irreps$(A_\hbar[\hbar^{-1}]) \leq \dim_{\mathbf{C}((\hbar))} \mathrm{HH}_0(A_\hbar[\hbar^{-1}]) \leq \dim \mathrm{HP}_0(\mathcal{O}_X)$. Fixed bound for all star-product quantizations!
- Better: F.d. irreps correspond to primitive ideals (the kernel of the irrep); the number supported at a given point $x \in X$ is at most $\dim \mathrm{HP}_0(\widehat{\mathcal{O}}_{X,x})$.
- This also bounds the number of prime ideals supported on a subvariety $Y \in X$, by taking a transverse slice Z to Y at any point $y \in Y$, and computing $\dim \mathrm{HP}_0(\widehat{\mathcal{O}}_{Z,y})$.
- In case $\mathcal{O}_X$ was graded, we can apply all this to filtered quantizations A.
- One has similar bounds on the number of (zero-dimensional) symplectic leaves of Poisson deformations of $\mathcal{O}_X$.

Symmetric powers of symplectic varieties (6)

Let Y be a connected affine symplectic variety (i.e., a smooth connected subvariety of $\mathbf{C}^m$ equipped with an algebraic symplectic form).

- $\mathrm{HP}_0(\mathcal{O}_Y) := \mathcal{O}_Y/\{\mathcal{O}_Y, \mathcal{O}_Y\} \cong H^{\dim_{\mathbf{C}} Y}(Y)$, via the isomorphism $[f] \mapsto [f \cdot \mathrm{vol}_Y] = [f \cdot \omega^{\dim_{\mathbf{C}} Y/2}]$.

Let $S^n Y = Y^n/S_n$ be the n-th symmetric power of Y for all $n \geq 1$.

Theorem

There is a canonical graded algebra isomorphism $(|\mathrm{HP}_0(\mathcal{O}_{S^n Y})^*| = n, |t| = 1 = |H^{\dim_{\mathbf{C}} Y}(Y)|)$:

$$\mathrm{Sym}(H^{\dim_{\mathbf{C}} Y}(Y)^*[t]) \xrightarrow{\sim} \bigoplus_{n \geq 0} \mathrm{HP}_0(\mathcal{O}_{S^n Y})^*,$$

$$\alpha \cdot t^{m-1} \mapsto \varphi_\alpha^{(m)}, \quad \varphi_\alpha^{(m)}(f) = \alpha(f^m \cdot \mathrm{vol}_Y).$$

Multiplication on RHS: $\varphi \cdot \psi = symm^*(\varphi \otimes \psi)$, $symm : \mathcal{O}_{S^{a+b} Y} \hookrightarrow \mathcal{O}_{S^a Y} \otimes \mathcal{O}_{S^b Y}, |\varphi| = a, |\psi| = b.$

The vector space case (7)

Next, take $Y = V$ a symplectic vector space.
Then, $S^n V \cong V \times (V^{n-1}/S_n)$, since we can average n points.

Theorem

$\mathrm{HP}_0(\mathcal{O}_{V^{n-1}/S_n}) \cong \mathbf{C}$.

Corollary

For arbitrary filtered or star-product quantizations A or $A_\hbar$ of $\mathcal{O}_{V^{n-1}/S_n}$, $\dim \mathrm{HH}_0(A) \leq 1$ and $\dim_{\mathbf{C}((\hbar))} \mathrm{HH}_0(A_\hbar[\hbar^{-1}]) \leq 1$, and the quantization admits at most one finite-dimensional irreducible representation.

For $A = \mathrm{Weyl}(V^{n-1})^{S_n}$ or more generally a spherical Cherednik algebra quantizing $\mathcal{O}_{V^{n-1}/S_n}$, we get an equality: $\dim HH_0(A) = 1$. Thus, the canonical surjection $\mathrm{HP}_0(\mathcal{O}_{V^{n-1}}^{S_n}) \twoheadrightarrow \mathrm{gr}\,\mathrm{HH}_0(A)$ is actually an isomorphism.

Idea of proof of theorems (8)

- Poisson traces $\mathsf{HP}_0(\mathcal{O}_X)^*$ are *algebraic distributions on X invariant under Hamiltonian flow.* Here Hamiltonian vector fields are $\xi_f := \{f, -\}$.
- For $X = S^n Y$ for Y symplectic, these are supported on diagonals of $S^n Y$.
- The problem reduces to the main diagonal: we need to identify distributions on the diagonal $Y \subseteq S^n Y$ invariant under Hamiltonian flow with $H^{\dim Y}(Y)$.**
- Equivalently, we need to identify S_n-invariant distributions on Y^n supported on the diagonal and invariant under the flow of S_n-invariant Hamiltonians with $H^{\dim Y}(Y)$.
- More generally, we can replace "distribution" by "D-module" (or compactly-supported C^∞ distribution) and $H^{\dim Y}(Y)$ by Ω_Y (or $H_c^{\dim Y}(Y)$).

**Cheating: we must also prove that the D-module is semisimple.

Idea of proof continued (9)

- This is now a local statement, so we can reduce to $Y = V = \mathbf{C}^{2m}$, and show that the space of symmetric polydifferential operators $\mathcal{O}_V^{\otimes n-1} \to \mathcal{O}_V$ invariant under flow of S_n-invariant Hamiltonians is one-dimensional.
- It is enough to consider the image of elements $f^{\otimes n-1}$ for $f \in \mathcal{O}_V$, since these span $\mathrm{Sym}^{n-1} \mathcal{O}_V$.
- Invariance under Hamiltonian flow implies invariance under symplectomorphisms of V.
- By the Darboux theorem, we can apply a symplectic (Poisson) automorphism of $\mathcal{O}_V$ taking f to any fixed (nonzero) linear function in V^*.
- By invariance under symplectomorphisms fixing f, the image of $f^{\otimes n-1}$ must be a polynomial in f.
- By invariance under symplectomorphisms rescaling f, the image must be $c \cdot f^{n-1}$ for some $c \in \mathbf{C}$.
- Hence, c uniquely determines the Poisson trace, so $\dim \mathrm{HP}_0(\mathcal{O}_{V^{n-1}/S_n})^* = 1$.

D-module statement (10)

- Given a Poisson variety X, let $M(X)$ denote the D-module $M(X) = D_X/(D_X\xi_f : f \in \mathcal{O}_X)$, where ξ_f is the Hamiltonian vector field associated to f.
- Its solutions are elements invariant under Hamiltonian flow.
- Let $\mathrm{HP}_i^{DR}(X) := \pi_i M(X)$ be the i-th derived pushforward under $\pi : X \to \mathrm{pt}$.

Theorem

- *For Y connected affine symplectic,*

$$M(S^n Y) \cong \bigoplus \Delta_*^\lambda(\Omega_Y \boxtimes \cdots \boxtimes \Omega_Y)^{Stab_{S_n}(\lambda)},$$

  summing over all partitions λ of n, with Δ^λ the corresponding diagonal map.
- $\mathrm{HP}_*^{DR}(S^\bullet Y) \cong \mathrm{Sym}^\bullet(H^{\dim Y - *}(Y)[t])$, $|t| = 1$ *in* $\bullet$ *grading.*

- If $\dim Y = 2$, $\mathrm{HP}_*^{DR}(S^n Y) \cong H^{2n-*}(\mathrm{Hilb}^n Y)$.
- In fact, $\rho_*\Omega_{\mathrm{Hilb}^n Y} \cong M(S^n Y)$, for $\rho : \mathrm{Hilb}^n Y \twoheadrightarrow S^n Y$.
- Conjecture: true for general symplectic resolutions $\rho : \widetilde{X} \to X$.

Type D_n singularities (11)

In contrast to type A_n (and $B_n = C_n$), the quotient $T^*\mathbf{C}^n/D_n$ does not admit a symplectic resolution for $\mathbf{C}^n$ the reflection rep of D_n.

Theorem

For type D_n, the surjection $\mathsf{HP}_0(\mathcal{O}_{\mathbf{C}^{2n}}^{D_n}) \twoheadrightarrow \operatorname{gr}\mathsf{HH}_0(\mathsf{Weyl}(\mathbf{C}^{2n})^{D_n})$ is only an isomorphism for $n \leq 6$.

The proof uses that $D_n < B_n$ of index two. Using a previous result for the $\cong$,

$$\bigoplus_{n\geq 0} \mathsf{HP}_0(\mathcal{O}_{\mathbf{C}^{2n}}^{D_n})^* \subseteq \bigoplus_{n\geq 0} \mathsf{HP}_0(\mathcal{O}_{\mathbf{C}^{2n}}^{B_n})^* \cong \mathsf{Sym}(\mathsf{HP}_0(\mathcal{O}_{\mathbf{C}^2}^{\mathbf{Z}/2})^*[t]).$$

Consider the latter as the polynomial functions on $\mathsf{HP}_0(\mathcal{O}_{\mathbf{C}^2}^{\mathbf{Z}/2})[[t]]$. Then I identify the LHS as the subspace annihilated by certain vector fields on $\mathsf{HP}_0(\mathcal{O}_{\mathbf{C}^2}^{\mathbf{Z}/2})[[t]]$.
By [Alev et al. '00], $\dim \mathsf{HH}_0(\mathsf{Weyl}(\mathbf{C}^{2n})^{D_n}) =$ number of partitions of n with an even # of parts. This is strictly smaller than the dimension of the above for $n \geq 7$.