Poisson traces on symmetric powers of symplectic varieties and type D_n Weyl group singularities (1)

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Based mostly on joint work with Etingof, to appear soon on arXiv
This talk is posted at `http://math.mit.edu/~trasched/`

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- Similarly, if $\mathcal{O}_X$ is a graded Poisson algebra, we can consider a filtered quantization: $A = \bigcup_{m\geq 0} A_{\leq m}$ is a filtered algebra, $\operatorname{gr} A := \bigoplus_{m\geq 0} A_m/A_{m-1} \cong \mathcal{O}_X$ ($A_{-1} := 0$), and $\{\operatorname{gr}_m a, \operatorname{gr}_n b\} = \operatorname{gr}_{m+n-1}[a,b]$ for $a \in A_{\leq m}, b \in A_{\leq n}$.

Goals and motivation (3)

- Main goal: describe $\mathrm{HP}_0(\mathcal{O}_X) := \mathcal{O}_X/\{\mathcal{O}_X, \mathcal{O}_X\}$ when X is the affine Poisson variety $X = S^n Y = Y^n/S_n$, for Y affine symplectic (and higher versions $\mathrm{HP}_i^{DR}(X), i > 0$).

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- Conjectures on symplectic resolutions
- Results on type D_n quotient singularities, i.e., $T^*\mathbf{C}^n/W$ with W a type D_n Weyl group acting on $\mathbf{C}^n$.

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- Proof: $[A_\hbar, A_\hbar] + \hbar^2 A_\hbar \supseteq \hbar\{\mathcal{O}_X, \mathcal{O}_X\} + \hbar^2 A_\hbar$.

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- One has similar bounds on the number of (zero-dimensional) symplectic leaves of Poisson deformations of $\mathcal{O}_X$.

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Let $S^n Y = Y^n/S_n$ be the n-th symmetric power of Y for all $n \geq 1$.

Theorem

There is a canonical graded algebra isomorphism $(|\mathrm{HP}_0(\mathcal{O}_{S^n Y})^*| = n, |t| = 1 = |H^{\dim_{\mathbf{C}} Y}(Y)|)$:

$$\mathrm{Sym}(H^{\dim_{\mathbf{C}} Y}(Y)^*[t]) \xrightarrow{\sim} \bigoplus_{n \geq 0} \mathrm{HP}_0(\mathcal{O}_{S^n Y})^*,$$

$$\alpha \cdot t^{m-1} \mapsto \varphi_\alpha^{(m)}, \quad \varphi_\alpha^{(m)}(f) = \alpha(f^m \cdot \mathrm{vol}_Y).$$

Multiplication on RHS: $\varphi \cdot \psi = \mathit{symm}^*(\varphi \otimes \psi)$, $\mathit{symm} : \mathcal{O}_{S^{a+b} Y} \hookrightarrow \mathcal{O}_{S^a Y} \otimes \mathcal{O}_{S^b Y}, |\varphi| = a, |\psi| = b$.

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For $A = \mathrm{Weyl}(V^{n-1})^{S_n}$ or more generally a spherical Cherednik algebra quantizing $\mathcal{O}_{V^{n-1}/S_n}$, we get an equality: $\dim HH_0(A) = 1$.
Thus, the canonical surjection $\mathrm{HP}_0(\mathcal{O}_{V^{n-1}}^{S_n}) \twoheadrightarrow \mathrm{gr}\, \mathrm{HH}_0(A)$ is actually an isomorphism.

Idea of proof of theorems (8)

- Poisson traces $\mathrm{HP}_0(\mathcal{O}_X)^*$ are *algebraic distributions on* X *invariant under Hamiltonian flow.* Here Hamiltonian vector fields are $\xi_f := \{f, -\}$.

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**Cheating: we must also prove that the *D*-module is semisimple.

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- Hence, c uniquely determines the Poisson trace, so $\dim \mathrm{HP}_0(\mathcal{O}_{V^{n-1}/S_n})^* = 1$.

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- In fact, $\rho_* \Omega_{\mathrm{Hilb}^n Y} \cong M(S^n Y)$, for $\rho : \mathrm{Hilb}^n Y \twoheadrightarrow S^n Y$.
- Conjecture: true for general symplectic resolutions $\rho : \widetilde{X} \to X$.

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Consider the latter as the polynomial functions on $\mathrm{HP}_0(\mathcal{O}_{\mathbf{C}^2}^{\mathbf{Z}/2})[[t]]$. Then I identify the LHS as the subspace annihilated by certain vector fields on $\mathrm{HP}_0(\mathcal{O}_{\mathbf{C}^2}^{\mathbf{Z}/2})[[t]]$.

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By [Alev et al. '00], $\dim \mathrm{HH}_0(\mathrm{Weyl}(\mathbf{C}^{2n})^{D_n}) =$ number of partitions of n with an even # of parts. This is strictly smaller than the dimension of the above for $n \geq 7$.