Poisson traces on symmetric powers of symplectic varieties and classical Weyl group singularities (1)

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Based mostly on joint work with Etingof, to appear soon on arXiv
This talk is posted at `http://math.mit.edu/~trasched/`

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- Conjectures on symmetric powers of isolated surface singularities.
- Core ingredient of proofs: study Hamiltonian-invariant distributions on X (more generally, a certain D-module on X).

Poisson and Hochschild traces (3)

Let A be an associative algebra and $\mathcal{O}_X$ a Poisson algebra.

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$\#$f.d. irreps$(A_\hbar[\hbar^{-1}]) \leq \dim_{\mathbf{C}((\hbar))} \mathrm{HH}_0(A_\hbar[\hbar^{-1}]) \leq \dim \mathrm{HP}_0(\mathcal{O}_X)$.
Fixed bound for all star-product quantizations!

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Theorem (Theorem A)

There is a canonical graded algebra isomorphism $(|\mathrm{HP}_0(\mathcal{O}_{S^n Y})^*| = n, |t| = 1 = |H^{\dim_{\mathbf{C}} Y}(Y)|)$*:*

$$\mathrm{Sym}(H^{\dim_{\mathbf{C}} Y}(Y)^*[t]) \xrightarrow{\sim} \bigoplus_{n \geq 0} \mathrm{HP}_0(\mathcal{O}_{S^n Y})^*,$$

$$\alpha \cdot t^{m-1} \mapsto \varphi_\alpha^{(m)}, \quad \varphi_\alpha^{(m)}(f) = \varphi_\alpha(f^m) = \alpha(f^m \cdot \mathrm{vol}_Y).$$

Corollaries (5)

Let $a_n(m)$ be the number of m-multipartitions of n, i.e., collections of m partitions $\lambda_1, \ldots, \lambda_m$ of numbers $n_1, \ldots, n_m$ such that $\sum_{i=1}^{m} n_i = n$.

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Corollary

Let $A_\hbar$ be any deformation quantization of $\mathcal{O}_{S^n Y}$. Then $\dim_{\mathbf{C}((\hbar))} \mathrm{HH}_0(A_\hbar[\hbar^{-1}]) \leq \dim \mathrm{HP}_0(\mathcal{O}_X) = a_n(\dim H^{\dim_{\mathbf{C}} Y}(Y))$.

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- New example: If Y is a surface with $H^1(Y) = 0$, then [EO] constructed the universal deformation $\widetilde{A}_\hbar$ of $\mathcal{O}_{Y^n} \rtimes S_n$. Then set $A_\hbar := e\widetilde{A}_\hbar e$, for $e = \frac{1}{n!}\sum_{g \in S_n} g \in \mathbf{C}[S_n]$ (a "global spherical symplectic reflection algebra").

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- Note: in all cases, $A_\hbar[\hbar^{-1}]$ has no finite-dimensional representations (since there are no points where all Hamiltonian flows vanish). However, using the next theorem, we can produce bounds on the number of prime ideals (which are actually independent of Y).

The vector space case (6)

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For arbitrary filtered or star-product quantizations A or $A_\hbar$ of $\mathcal{O}_{V^{n-1}/S_n}$, $\dim \mathrm{HH}_0(A) \leq 1$ and $\dim_{\mathbf{C}((\hbar))} \mathrm{HH}_0(A_\hbar[\hbar^{-1}]) \leq 1$, and the quantization admits at most one finite-dimensional irreducible representation.

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For $A = \mathrm{Weyl}(V^{n-1})^{S_n}$, the S_n-invariant elements of the Weyl algebra of V^{n-1}, using [Alev et al '00], we deduce $\dim \mathrm{HH}_0(A) = 1$, i.e., the canonical surjection $\mathrm{HP}_0(\mathcal{O}_{V^{n-1}}^{S_n}) \twoheadrightarrow \mathrm{gr}\, \mathrm{HH}_0(A)$ is actually an isomorphism.

Singularities associated to classical Weyl groups (7)

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Theorem (Etingof-S. '09b)

For types $B_n = C_n = S_n \rtimes (\mathbf{Z}/2)^n$,

$$\mathrm{HP}_0(\mathcal{O}_{\mathbf{C}^{2n}}^{B_n}) \xrightarrow{\sim} \operatorname{gr} \mathrm{HH}_0(A), \quad A = \mathrm{Weyl}(\mathbf{C}^{2n})^{B_n}.$$

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The same is true if B_n *is replaced by* $G_n := S_n \rtimes G^n$ *for arbitrary finite* $G < \mathrm{SL}_2(\mathbf{C})$.

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- Explicitly, we showed $\bigoplus_n \mathrm{HP}_0(\mathcal{O}_{\mathbf{C}^{2n}}^{G_n})^* \cong \mathrm{Sym}(\mathrm{HP}_0(\mathcal{O}_{\mathbf{C}^2}^{G})^*[t])$ for *all* finite $G < \mathrm{SL}_2(\mathbf{C})$ [noncanonically!].

Singularities associated to classical Weyl groups (7)

- When $V = \mathbf{C}^2$, $V^{n-1} = \mathbf{C}^{n-1} \oplus \mathbf{C}^{n-1} = T^*\mathbf{C}^{n-1}$;
  $\mathbf{C}^{n-1}$ = reflection rep of S_n, the type A_{n-1} Weyl group.

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- The theorems extend to the case where A is an arbitrary spherical symplectic reflection algebra (SSRA) quantizing $\mathcal{O}_{\mathbf{C}^{2n}}^{G_n}$ or $\mathcal{O}_{\mathbf{C}^{2n-2}}^{S_n}$.
  (i.e., rational Cherednik in cases $B_n = C_n, G = \mathbf{Z}/m, A_{n-1}$).

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Theorem (Theorem C)

For type D_n, the surjection $\mathrm{HP}_0(\mathcal{O}_{\mathbf{C}^{2n}}^{D_n}) \twoheadrightarrow \mathrm{gr}\,\mathrm{HH}_0(\mathrm{Weyl}(\mathbf{C}^{2n})^{D_n})$ is only an isomorphism for $n \leq 6$.

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Consider the latter as the polynomial functions on $\mathrm{HP}_0(\mathcal{O}_{\mathbf{C}^2}^{\mathbf{Z}/2})[[t]]$. Then I identify the LHS as the subspace annihilated by certain vector fields on $\mathrm{HP}_0(\mathcal{O}_{\mathbf{C}^2}^{\mathbf{Z}/2})[[t]]$.

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By [Alev et al. '00], $\dim \mathrm{HH}_0(\mathrm{Weyl}(\mathbf{C}^{2n})^{D_n})$ = number of partitions of n with an even # of parts. This is strictly smaller than the dimension of the above for $n \geq 7$.

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- When $\dim Y = 2$, we deduce $\mathrm{HP}_0(\mathcal{O}_{S^n Y}) \cong H^{2n}(\mathrm{Hilb}^n Y)$, where $\mathrm{Hilb}^n Y$ is the (smooth) Hilbert scheme of n points of Y, a symplectic resolution of $S^n Y$.

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("Geometric" = global sections of a quantization of $\mathcal{O}_{\widetilde{X}}$; e.g., SSRAs such as $\mathrm{Weyl}(\mathbf{C}^{2n})^{G_n}$ when $X = \mathbf{C}^{2n}/G_n$.)

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We proved (a) ([Etingof-S. '09b]) and [Alev-Lambre '98] proved (b) when $Z \subseteq \mathbf{C}^3$ is quasihomogeneous.

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**Cheating: we must also prove that the D-module is semisimple.

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- The result is one-dimensional.