

Poisson traces on symmetric powers of symplectic varieties and classical Weyl group singularities (1)

Travis Schedler (AIM/MIT)

Tue, Jun 7, 2011, U. Notre Dame

Based mostly on joint work with Etingof, to appear soon on arXiv

This talk is posted at <http://math.mit.edu/~trasched/>

Goals and motivation (2)

- Goal: compute $\text{HP}_0(\mathcal{O}_X) := \mathcal{O}_X / \{\mathcal{O}_X, \mathcal{O}_X\}$ where X is one of the following singular affine (complex) Poisson varieties:
 - (A) $X = S^n Y := Y^n / S_n$, where Y is a symplectic variety;
 - (B) \mathbf{C}^{2n-2} / S_n , where $\mathbf{C}^{2n-2} = \mathbf{C}^{n-1} \oplus \mathbf{C}^{n-1} = T^* \mathbf{C}^{n-1}$, $\mathbf{C}^{n-1} =$ reflection representation of S_n (type A_{n-1});
 - (C) \mathbf{C}^{2n} / H , where H is a type $B_n = C_n$ or D_n Weyl group acting on $\mathbf{C}^{2n} = \mathbf{C}^n \oplus \mathbf{C}^n$.
- Deduce consequences for *all* quantizations of these varieties: bounds on Hochschild homology ($\dim \text{HH}_0$), and on numbers of finite-dimensional irreps (and prime ideals).
- Conjectures on symplectic resolutions, confirmed by (A) (when $\dim_{\mathbf{C}} Y = 2$) and (B) (and previous results).
- Conjectures on symmetric powers of isolated surface singularities.
- Core ingredient of proofs: study Hamiltonian-invariant distributions on X (more generally, a certain D -module on X).

Poisson and Hochschild traces (3)

Let A be an associative algebra and \mathcal{O}_X a Poisson algebra.

Definition 1. $\text{HH}_0(A) := A/[A, A]$ “zeroth Hochschild homology;” $\text{HH}_0(A)^* = \{\varphi : A \rightarrow \mathbf{C} \mid \varphi([a, b]) = 0, \forall a, b\}$ “Hochschild traces.”

- If $\rho : A \rightarrow \text{End}(V)$ is a f.d. rep, then $\text{tr}(\rho) \in \text{HH}_0(A)^*$.

- Standard theorem: If ρ_1, \dots, ρ_n distinct f.d. irreps, then $\text{tr}(\rho_1), \dots, \text{tr}(\rho_n) \in \text{HH}_0(A)^*$ are lin. independent.
- Corollary: $\#\text{f.d. irreps}(A) \leq \dim \text{HH}_0(A)$.

Definition 2. $\text{HP}_0(\mathcal{O}_X) := \mathcal{O}_X / \{\mathcal{O}_X, \mathcal{O}_X\}$ “zeroth Poisson homology.” $\text{HP}_0(\mathcal{O}_X)^* = \{\varphi : \mathcal{O}_X \rightarrow \mathbf{C} \mid \varphi|_{\{\mathcal{O}_X, \mathcal{O}_X\}} = 0\}$ “Poisson traces.”

- Proposition: If A_{\hbar} is a star-product quantization of \mathcal{O}_X , then $\dim_{\mathbf{C}((\hbar))} \text{HH}_0(A[\hbar^{-1}]) \leq \dim \text{HP}_0(\mathcal{O}_X)$.
- Proof: $[A_{\hbar}, A_{\hbar}] + \hbar^2 A_{\hbar} \supseteq \hbar \{\mathcal{O}_X, \mathcal{O}_X\}$. We deduce:

$\#\text{f.d. irreps}(A_{\hbar}[\hbar^{-1}]) \leq \dim_{\mathbf{C}((\hbar))} \text{HH}_0(A_{\hbar}[\hbar^{-1}]) \leq \dim \text{HP}_0(\mathcal{O}_X)$. Fixed bound for all star-product quantizations!

Symmetric powers of symplectic varieties (4)

Let Y be a connected affine symplectic variety (i.e., a smooth connected subvariety of \mathbf{C}^m equipped with an algebraic symplectic form).

- $\text{HP}_0(\mathcal{O}_Y) := \mathcal{O}_Y / \{\mathcal{O}_Y, \mathcal{O}_Y\} \cong H^{\dim_{\mathbf{C}} Y}(Y)$, via the isomorphism $[f] \mapsto [f \cdot \text{vol}_Y] = [f \cdot \omega^{\dim_{\mathbf{C}} Y/2}]$.
- Dually, $H^{\dim_{\mathbf{C}} Y}(Y)^* \xrightarrow{\sim} \text{HP}_0(\mathcal{O}_Y)^*$, $\alpha \mapsto \varphi_{\alpha}$, $\varphi_{\alpha}([f]) = \alpha([f \cdot \text{vol}_Y])$.

Let $S^n Y = Y^n / S_n$ be the n -th symmetric power of Y for all $n \geq 1$.

Theorem 3 (Theorem A). *There is a canonical graded algebra isomorphism $(|\text{HP}_0(\mathcal{O}_{S^n Y})^*| = n, |t| = 1 = |H^{\dim_{\mathbf{C}} Y}(Y)|)$:*

$$\text{Sym}(H^{\dim_{\mathbf{C}} Y}(Y)^*[t]) \xrightarrow{\sim} \bigoplus_{n \geq 0} \text{HP}_0(\mathcal{O}_{S^n Y})^*,$$

$$\alpha \cdot t^{m-1} \mapsto \varphi_{\alpha}^{(m)}, \quad \varphi_{\alpha}^{(m)}(f) = \varphi_{\alpha}(f^m) = \alpha(f^m \cdot \text{vol}_Y).$$

Corollaries (5)

Let $a_n(m)$ be the number of m -multipartitions of n , i.e., collections of m partitions $\lambda_1, \dots, \lambda_m$ of numbers n_1, \dots, n_m such that $\sum_{i=1}^m n_i = n$.

Corollary 4. *Let A_{\hbar} be any deformation quantization of $\mathcal{O}_{S^n Y}$. Then $\dim_{\mathbf{C}((\hbar))} \text{HH}_0(A_{\hbar}[\hbar^{-1}]) \leq \dim \text{HP}_0(\mathcal{O}_X) = a_n(\dim H^{\dim_{\mathbf{C}} Y}(Y))$.*

- Equality holds if $A_{\hbar} = \text{Sym}^n B_{\hbar}$ for B_{\hbar} a quantization of \mathcal{O}_Y , by a result of Etingof-Oblomkov '03.
- New example: If Y is a surface with $H^1(Y) = 0$, then [EO] constructed the universal deformation \tilde{A}_{\hbar} of $\mathcal{O}_{Y^n} \rtimes S_n$. Then set $A_{\hbar} := e\tilde{A}_{\hbar}e$, for $e = \frac{1}{n!} \sum_{g \in S_n} g \in \mathbf{C}[S_n]$ (a “global spherical symplectic reflection algebra”).
- Note: in all cases, $A_{\hbar}[\hbar^{-1}]$ has no finite-dimensional representations (since there are no points where all Hamiltonian flows vanish). However, using the next theorem, we can produce bounds on the number of prime ideals (which are actually independent of Y).

The vector space case (6)

Next, take $Y = V$ a symplectic vector space. Then, $S^n V \cong V \times (V^{n-1}/S_n)$, since we can average n points.

Theorem 5 (Theorem B). $\text{HP}_0(\mathcal{O}_{V^{n-1}/S_n}) \cong \mathbf{C}$.

Corollary 6. *For arbitrary filtered or star-product quantizations A or A_{\hbar} of $\mathcal{O}_{V^{n-1}/S_n}$, $\dim \text{HH}_0(A) \leq 1$ and $\dim_{\mathbf{C}((\hbar))} \text{HH}_0(A_{\hbar}[[\hbar^{-1}]]) \leq 1$, and the quantization admits at most one finite-dimensional irreducible representation.*

For $A = \text{Weyl}(V^{n-1})^{S_n}$, the S_n -invariant elements of the Weyl algebra of V^{n-1} , using [Alev et al '00], we deduce $\dim \text{HH}_0(A) = 1$, i.e., the canonical surjection $\text{HP}_0(\mathcal{O}_{V^{n-1}}^{S_n}) \rightarrow \text{gr HH}_0(A)$ is actually an isomorphism.

Singularities associated to classical Weyl groups (7)

- When $V = \mathbf{C}^2$, $V^{n-1} = \mathbf{C}^{n-1} \oplus \mathbf{C}^{n-1} = T^*\mathbf{C}^{n-1}$; $\mathbf{C}^{n-1} =$ reflection rep of S_n , the type A_{n-1} Weyl group.

Theorem 7 (Etingof-S. '09b). *For types $B_n = C_n = S_n \times (\mathbf{Z}/2)^n$,*

$$\text{HP}_0(\mathcal{O}_{\mathbf{C}^{2n}}^{B_n}) \xrightarrow{\sim} \text{gr HH}_0(A), \quad A = \text{Weyl}(\mathbf{C}^{2n})^{B_n}.$$

The same is true if B_n is replaced by $G_n := S_n \times G^n$ for arbitrary finite $G < \text{SL}_2(\mathbf{C})$.

- Explicitly, we showed $\bigoplus_n \text{HP}_0(\mathcal{O}_{\mathbf{C}^{2n}}^{G_n})^* \cong \text{Sym}(\text{HP}_0(\mathcal{O}_{\mathbf{C}^2}^G)^*[t])$ for all finite $G < \text{SL}_2(\mathbf{C})$ [noncanonically!].
- The theorems extend to the case where A is an arbitrary spherical symplectic reflection algebra (SSRA) quantizing $\mathcal{O}_{\mathbf{C}^{2n}}^{G_n}$ or $\mathcal{O}_{\mathbf{C}^{2n-2}}^{S_n}$. (i.e., rational Cherednik in cases $B_n = C_n, G = \mathbf{Z}/m, A_{n-1}$).

Type D_n singularities (8)

In contrast:

Theorem 8 (Theorem C). *For type D_n , the surjection $\text{HP}_0(\mathcal{O}_{\mathbf{C}^{2n}}^{D_n}) \rightarrow \text{gr HH}_0(\text{Weyl}(\mathbf{C}^{2n})^{D_n})$ is only an isomorphism for $n \leq 6$.*

The proof uses that $D_n < B_n$ of index two. Using the previous theorem for the \cong ,

$$\bigoplus_{n \geq 0} \text{HP}_0(\mathcal{O}_{\mathbf{C}^{2n}}^{D_n})^* \subseteq \bigoplus_{n \geq 0} \text{HP}_0(\mathcal{O}_{\mathbf{C}^{2n}}^{B_n})^* \cong \text{Sym}(\text{HP}_0(\mathcal{O}_{\mathbf{C}^2}^{\mathbf{Z}/2})^*[t]).$$

Consider the latter as the polynomial functions on $\text{HP}_0(\mathcal{O}_{\mathbf{C}^2}^{\mathbf{Z}/2})[[t]]$. Then I identify the LHS as the subspace annihilated by certain vector fields on $\text{HP}_0(\mathcal{O}_{\mathbf{C}^2}^{\mathbf{Z}/2})[[t]]$. By [Alev et al. '00], $\dim \text{HH}_0(\text{Weyl}(\mathbf{C}^{2n})^{D_n}) =$ number of partitions of n with an even # of parts. This is strictly smaller than the dimension of the above for $n \geq 7$.

Conjectures on symplectic resolutions (9)

- When $\dim Y = 2$, we deduce $\mathrm{HP}_0(\mathcal{O}_{S^n Y}) \cong H^{2n}(\mathrm{Hilb}^n Y)$, where $\mathrm{Hilb}^n Y$ is the (smooth) Hilbert scheme of n points of Y , a symplectic resolution of $S^n Y$.
- Also, if Y is the minimal resolution of \mathbf{C}^2/G for $G < \mathrm{SL}_2(\mathbf{C})$, then $\mathrm{HP}_0(\mathcal{O}_{\mathbf{C}^{2n}/G_n}) \cong H^{2n}(\mathrm{Hilb}^n Y)$, and again $\mathrm{Hilb}^n Y \rightarrow \mathbf{C}^{2n}/G_n = S^n(\mathbf{C}^2/G)$ is a symplectic resolution.
- On the other hand, note that \mathbf{C}^{2n}/D_n does *not* admit a symplectic resolution (by [Bellamy '07]).

Conjecture 0.1. *Suppose $\tilde{X} \rightarrow X$ is a symplectic resolution of an affine Poisson variety X .*

- $\mathrm{HP}_0(\mathcal{O}_X) \cong H^{\dim_{\mathbf{C}} X}(\tilde{X})$.
- If A_{\hbar} is a “geometric” star-product quantization of \mathcal{O}_X , then $\dim \mathrm{HP}_0(\mathcal{O}_X) = \dim_{\mathbf{C}(\hbar)} \mathrm{HH}_0(A_{\hbar}[\hbar^{-1}])$.

(“Geometric” = global sections of a quantization of $\mathcal{O}_{\tilde{X}}$; e.g., SSRAs such as $\mathrm{Weyl}(\mathbf{C}^{2n})^{G_n}$ when $X = \mathbf{C}^{2n}/G_n$.)

Conjectures on isolated surface singularities (10)

Now we let Z be an affine Poisson surface ($\dim. 2$) which is generically symplectic with isolated singularities.

Conjecture 0.2. (a) *One has a (noncanonical!) graded algebra isomorphism*

$$\bigoplus_{n \geq 0} \mathrm{HP}_0(\mathcal{O}_{S^n Z})^* \cong \mathrm{Sym}(\mathrm{HP}_0(\mathcal{O}_Z)^*[t]).$$

(b) *Let z_1, \dots, z_m be the singularities of Z with Milnor numbers μ_1, \dots, μ_m . Then,*

$$\mathrm{HP}_0(\mathcal{O}_Z) \cong H^2(Z) \oplus \bigoplus_{i=1}^m \mathbf{C}^{\mu_i}.$$

We proved (a) ([Etingof-S. '09b]) and [Alev-Lambre '98] proved (b) when $Z \subseteq \mathbf{C}^3$ is quasihomogeneous.

Idea of proof of Theorem A (11)

- Poisson traces $\mathrm{HP}_0(\mathcal{O}_X)^*$ are *algebraic distributions on X invariant under Hamiltonian flow*. Here Hamiltonian vector fields are $\xi_f := \{f, -\}$.
- For $X = S^n Y$ for Y symplectic, these are supported on diagonals of $S^n Y$.

- The problem reduces to the main diagonal: we need to identify distributions on the diagonal $Y \subseteq S^n Y$ invariant under Hamiltonian flow with $H^{\dim Y}(Y)$. **
- Equivalently, we need to identify S_n -invariant distributions on Y^n supported on the diagonal and invariant under the flow of S_n -invariant Hamiltonians with $H^{\dim Y}(Y)$.
- More generally, we can replace “distribution” by “ D -module” (or compactly-supported C^∞ distribution) and $H^{\dim Y}(Y)$ by Ω_Y (or $H_c^{\dim Y}(Y)$).

**Cheating: we must also prove that the D -module is semisimple.

Idea of proof of Theorem A, continued (12)

- This is now a local statement, so we can reduce to $Y = V = \mathbf{C}^{2m}$, and show that the space of symmetric polydifferential operators $\mathcal{O}_V^{\otimes n-1} \rightarrow \mathcal{O}_V$ invariant under flow of S_n -invariant Hamiltonians is one-dimensional.
- It is enough to consider the image of elements $f^{\otimes n-1}$ for $f \in \mathcal{O}_V$, since these span $\text{Sym}^{n-1} \mathcal{O}_V$.
- Invariance under Hamiltonian flow implies invariance under symplectomorphisms of V .
- By the Darboux theorem, it is enough to restrict to a linear function $f \in V^*$.
- By invariance under symplectomorphisms fixing f , the image of $f^{\otimes n-1}$ must be a polynomial in f .
- By invariance under symplectomorphisms rescaling f , the image must be $c \cdot f^{n-1}$ for some $c \in \mathbf{C}$.
- The result is one-dimensional.