Computational approaches to Poisson traces associated to finite subgroups of $\mathrm{Sp}_{2n}(\mathbf{C})$ (1)

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Based on joint work with Etingof, Gong, Pacchiano, and Ren
(arXiv:1101.5171)

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Main goal: Let V be a symplectic vector space and $G < \mathrm{Sp}(V)$ a finite subgroup. I will reduce the computation of $\mathcal{O}_V^G/\{\mathcal{O}_V^G, \mathcal{O}_V^G\}$ and $\mathcal{O}_V/\{\mathcal{O}_V^G, \mathcal{O}_V\}$ to a finite one, and do it by computer for many V and G.

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I. Brief introduction to Poisson varieties and Poisson traces
II. Some computational results
III. Finite-dimensionality of Poisson traces on V/G (i.e., of the main objects of study)
IV. A lemma on regular sequences and a bound on dimension
V. A bound on degree for $G < \mathrm{GL}_n < \mathrm{Sp}_{2n}$
VI. A general bound on the Hilbert series

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- A *filtered* noncommutative deformation, i.e., $A = \bigcup_m A_{\leq m}$ an associative (noncommutative) algebra such that $\operatorname{gr} A := \bigoplus_m A_{\leq m}/A_{\leq (m-1)} \cong \mathcal{O}_X$.

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- Then obtain $\{-,-\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$, from $\{\operatorname{gr}_m a, \operatorname{gr}_n b\} := \operatorname{gr}_{m+n-d} [a, b]$ (for all $a \in A_{\leq m}, b \in A_{\leq n}$; fixed $d \geq 1$).

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- Straightforward: $\{-,-\}$ is a Lie bracket satisfying $\{fg, h\} = f\{g, h\} + g\{f, h\}$. Called *Poisson bracket*.
- Conversely, beginning with a Poisson algebra $\mathcal{O}_X$, a *quantization* of $\mathcal{O}_X$ (or the affine variety X) is an filtered algebra A such that $\operatorname{gr} A = \mathcal{O}_X$ as Poisson algebras.

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  (More general quantization: spherical symplectic reflection algebras)

Poisson and Hochschild traces (5)

Let A be an associative algebra and $\mathcal{O}_X$ a Poisson algebra.

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$\mathrm{HH}_0(A) := A/[A, A]$ "zeroth Hochschild homology;"
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End of Part I.

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- But first, I will give some computational results that use this and computer programs.

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- For G-invariants ($\mathsf{HP}_0(\mathcal{O}_V^G) \cong \mathsf{HH}_0(A^G)$) in the case of Weyl groups of rank ≤ 2 this is due to Alev and Foissy; for Weyl groups of rank ≤ 3 it is due to Butin.

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- We compute in all of the above cases also the *grading* on HP_0 and hence $\mathrm{gr}\,\mathrm{HH}_0$, as well as the G-structure.

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- We compute in all of the above cases also the *grading* on HP_0 and hence $\mathrm{gr}\,\mathrm{HH}_0$, as well as the G-structure.
- In other joint work, I computed $h(\mathrm{HP}_0(\mathcal{O}_V^G); t)$ for types $B = C$ (more generally $H^n \rtimes S_n$ for $H < \mathrm{SL}_2(\mathbf{C})$), and in preparation are types A and D. The isomorphism $\mathrm{HP}_0(\mathcal{O}_V^G) \cong \mathrm{HH}_0(A^G)$ holds in types A, B, C but *not* for $D_n, n \geq 7$.

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- These were classified by Shephard and Todd, and fall into the infinite families S_{n+1} and $G(m, p, n)$ (rank n, i.e., in GL_n), along with 34 exceptional groups, $G_4, G_5, \ldots, G_{37}$. The ones $G_4, \ldots, G_{22}$ have rank two (are in GL_2), along with $G(m, p, 2)$.

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Remark: In related joint work, I showed that $\mathsf{HP}_0(\mathcal{O}_X, \mathcal{O}_Y)$ is finite-dimensional when X has finitely many symplectic leaves and $X \to Y$ is a finite morphism (X is an affine Poisson variety and Y an affine variety). The proof used D-modules.

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- Explicitly, $\xi_h^* : \mathcal{O}_{V^*} \to \mathcal{O}_{V^*}$ is a differential operator, the Fourier transform of ξ_h: namely, $\xi_h^* = F(\xi_h)$, $F : \mathcal{D}_V \xrightarrow{\sim} \mathcal{D}_{V^*}^{\mathrm{op}}$ is the algebra anti-isomorphism sending multiplication by a linear function $u \in V^*$ to $\partial_u \in \mathcal{D}_{V^*}$, and differentiation $\partial_v, v \in V$ to multiplication by v.

Continuation of proof (11)

- Now, fix $u \in V^*$. For $\eta \in \mathcal{D}_{V^*}$, let $\eta|_u$ be obtained by evaluating the coefficients of η at u: i.e., $\eta|_u \in \operatorname{Sym} V^* = \mathbf{C}[\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}]$ for a basis $x_1, \ldots, x_n, y_1, \ldots, y_n$ of V^*.

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End of part III.

IV. A lemma on regular sequences (12)

Let U be an arbitrary finite-dimensional vector space, and set $m = \dim U$. Recall that a regular sequence $h_1, \ldots, h_k \in \mathcal{O}_U$ is one such that h_i is not a zero divisor in $\mathcal{O}_U/(h_1, \ldots, h_{i-1})$, i.e., the zero locus of $h_1, \ldots, h_k$ has codimension k.

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- Recall that $R_u = \mathrm{Sym}\, V^*/(\xi_h^*|_u : h \in \mathcal{O}_V^G)$.
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V. A bound on degree for $G < \mathrm{GL}_n < \mathrm{Sp}_{2n}$: Hilbert series bound (16)

- For $G < \mathrm{GL}_n$, write $V = U \oplus W$, $\mathcal{O}_U = \mathbf{C}[x_1, \ldots, x_n]$ and $\mathcal{O}_W = \mathbf{C}[y_1, \ldots, y_n]$, with U and W Lagrangian, and $(x_i, y_j) = \delta_{ij}$.

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- Hence, the top degree of R_u in W^* equals that of $\bar{R}_u$.
- If $g_1, \ldots, g_n$ is a regular sequence in $\mathbf{C}[W]$, we conclude, for generic $u \in U^*$:

$$\mathrm{topdeg}(\mathrm{HP}_0(cO_V^G, \mathcal{O}_V)) \leq 2\,\mathrm{topdeg}_{W^*}(\overline{R_u}) \leq 2\sum_i (|g_i| - 2).$$

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$G(m,p,n)$, $m,n>1$: $n(n-1)m+2mn/p-4n$

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G_9:	56	G_{10}:	64	G_{11}:	88	G_{12} :	20	G_{13}:	32
G_{14}:	52	G_{15}:	64	G_{16}	92	G_{17}:	152	G_{18}:	172
G_{19}:	232	G_{20}:	76	G_{21}:	136	G_{22}:	56	G_{23}:	24
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- Then $(F_1(f)|_{\mathbf{C}\cdot u}, \ldots, F_N(f)|_{\mathbf{C}\cdot u})$ is an eigenvector of $B_\xi := C_\xi - \mathrm{Diag}(|F_1|, \ldots, |F_n|)$ of eigenvalue $|f|$.

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- Let $C_\xi \in \mathrm{Mat}_N(\mathbf{C})$ be such that $(\xi F_1, \ldots, \xi F_n)(f)(u) = \big(C_\xi(F_1, \ldots, F_n)\big)(f)(u)$, $\forall f \in \mathrm{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)^*$.
- Then $(F_1(f)|_{\mathbf{C}\cdot u}, \ldots, F_N(f)|_{\mathbf{C}\cdot u})$ is an eigenvector of $B_\xi := C_\xi - \mathrm{Diag}(|F_1|, \ldots, |F_n|)$ of eigenvalue $|f|$.
- Conclusion: $h(\mathrm{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)^*; t) \leq \sum_i \dim E_i(B_\xi)$, where E_i is the eigenspace of eigenvalue i.

The bound on $h(\mathrm{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V); t)$ continued (20)

- Recall: For every $\phi \in \mathcal{D}_{V^*}$, there exists $\psi = \sum_i \lambda_i F_i$, $\lambda_i \in \mathbf{C}$ such that, for all $f \in \mathrm{HP}_0(\mathcal{O}_V, \mathcal{O}_V^G)^*$, $\phi(f)(u) = \psi(f)(u)$.
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- Let $C_\xi \in \mathrm{Mat}_N(\mathbf{C})$ be such that $(\xi F_1, \ldots, \xi F_n)(f)(u) = \big(C_\xi(F_1, \ldots, F_n)\big)(f)(u)$, $\forall f \in \mathrm{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)^*$.
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- Conclusion: $h(\mathrm{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)^*; t) \leq \sum_i \dim E_i(B_\xi)$, where E_i is the eigenspace of eigenvalue i.
- This very often gives exactly the correct top degree!

The bound on $h(\mathrm{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V); t)$ continued (20)

- Recall: For every $\phi \in \mathcal{D}_{V^*}$, there exists $\psi = \sum_i \lambda_i F_i$, $\lambda_i \in \mathbf{C}$ such that, for all $f \in \mathrm{HP}_0(\mathcal{O}_V, \mathcal{O}_V^G)^*$, $\phi(f)(u) = \psi(f)(u)$.
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- Conclusion: $h(\mathrm{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)^*; t) \leq \sum_i \dim E_i(B_\xi)$, where E_i is the eigenspace of eigenvalue i.
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- Can combine this with the bound in the GL_n case: using the GL_n bound we can compute the above matrix modulo p, for p greater than the bound given by the GL_n argument.

The bound on $h(\mathrm{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V); t)$ continued (20)

- Recall: For every $\phi \in \mathcal{D}_{V^*}$, there exists $\psi = \sum_i \lambda_i F_i$, $\lambda_i \in \mathbf{C}$ such that, for all $f \in \mathrm{HP}_0(\mathcal{O}_V, \mathcal{O}_V^G)^*$, $\phi(f)(u) = \psi(f)(u)$.
- Using this claim: let $\xi \in \mathcal{D}_{V^*}$ be the Euler vector field: $[\xi, \eta] = |\eta|\eta$; $|v| = 1$ and $|\partial_w| = -1$ for $v \in V$ and $w \in V^*$.
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End of talk.