Computational approaches to Poisson traces associated to finite subgroups of $\mathsf{Sp}_{2n}(\mathbf{C})$ (1)

Travis Schedler (AIM/MIT)

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Main goal and outline (2)

Main goal: Let V be a symplectic vector space and $G < \mathsf{Sp}(V)$ a finite subgroup. I will reduce the computation of $\mathcal{O}_V^G/\{\mathcal{O}_V^G, \mathcal{O}_V^G\}$ and $\mathcal{O}_V/\{\mathcal{O}_V^G, \mathcal{O}_V\}$ to a finite one, and do it by computer for many V and G.

I. Brief introduction to Poisson varieties and Poisson traces

II. Some computational results

III. Finite-dimensionality of Poisson traces on V/G (i.e., of the main objects of study)

IV. A lemma on regular sequences and a bound on dimension

V. A bound on degree for $G < \mathsf{GL}_n < \mathsf{Sp}_{2n}$

VI. A general bound on the Hilbert series

Part I: Introduction to Poisson varieties and Poisson traces Recollections on quantization(3)

- Let $X \subseteq \mathbf{C}^N$ (affine), $\mathcal{O}_X =$ global functions on X.
- Suppose $\mathcal{O}_X = \bigoplus_{m \geq 0} (\mathcal{O}_X)_m$ is graded (X conical).
- A *filtered* noncommutative deformation, i.e., $A = \bigcup_m A_{\leq m}$ an associative (noncommutative) algebra such that $\mathsf{gr}\, A := \bigoplus_m A_{\leq m}/A_{\leq (m-1)} \cong \mathcal{O}_X$.
- Then obtain $\{-,-\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$, from $\{\mathsf{gr}_m\, a, \mathsf{gr}_n\, b\} := \mathsf{gr}_{m+n-d}[a,b]$ (for all $a \in A_{\leq m}, b \in A_{\leq n}$; fixed $d \geq 1$).

- Straightforward: $\{-,-\}$ is a Lie bracket satisfying $\{fg,h\} = f\{g,h\} + g\{f,h\}$. Called *Poisson bracket.*

- Conversely, beginning with a Poisson algebra $\mathcal{O}_X$, a *quantization* of $\mathcal{O}_X$ (or the affine variety X) is an filtered algebra A such that $\mathsf{gr}\, A = \mathcal{O}_X$ as Poisson algebras.

Examples related to subgroups of Sp_{2n} (4)

Example 1.
- $X = \mathbf{C}^{2n}$, $\mathcal{O}_X = \mathbf{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $\{f,g\} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}$.

- Quantization: Weyl algebra $\mathsf{Weyl}(\mathbf{C}^{2n})$ = algebra generated by x_i, y_i with relations $[x_i, y_j] = \delta_{ij}$, $[x_i, x_j] = 0 = [y_i, y_j]$. $(d = 2)$

Note: $\mathsf{Weyl}(\mathbf{C}^{2n}) \cong$ ring of polynomial differential operators on $\mathbf{C}^n$.

Example 2.
- If $G < \mathsf{Sp}_{2n}$, then $X = \mathbf{C}^{2n}/G$, $\mathcal{O}_X := \mathcal{O}_{C^{2n}}^G$ is Poisson;

- One quantization: $\mathsf{Weyl}(\mathbf{C}^{2n})^G$. (More general quantization: spherical symplectic reflection algebras)

Poisson and Hochschild traces (5)

Let A be an associative algebra and $\mathcal{O}_X$ a Poisson algebra.

Definition 3. $\mathsf{HH}_0(A) := A/[A,A]$ "zeroth Hochschild homology;" $\mathsf{HH}_0(A)^* = \{\varphi : A \to \mathbf{C} \mid \varphi([a,b]) = 0, \forall a, b\}$ "Hochschild traces."

- If $\rho : A \to \mathrm{End}(V)$ is a f.d. rep, then $\mathrm{tr}(\rho) \in \mathsf{HH}_0(A)^*$.

- Standard theorem: If $\rho_1, \ldots, \rho_n$ distinct f.d. irreps, then $\mathrm{tr}(\rho_1), \ldots, \mathrm{tr}(\rho_n) \in \mathsf{HH}_0(A)^*$ are lin. independent.

- Corollary: #f.d. irreps$(A) \leq \dim \mathsf{HH}_0(A)$.

Definition 4. $\mathsf{HP}_0(\mathcal{O}_X) := \mathcal{O}_X/\{\mathcal{O}_X, \mathcal{O}_X\}$ "zeroth Poisson homology." $\mathsf{HP}_0(\mathcal{O}_X)^* = \{\varphi : \mathcal{O}_X \to \mathbf{C} \mid \varphi|_{\{\mathcal{O}_X, \mathcal{O}_X\}} = 0\}$ "Poisson traces."

- Proposition: If $\mathsf{gr}\, A = \mathcal{O}_X$, then $\mathsf{HP}_0(\mathcal{O}_X) \twoheadrightarrow \mathsf{gr}\, \mathsf{HH}_0(A)$.

- Proof: $\mathsf{gr}[A,A] \supseteq \{\mathcal{O}_X, \mathcal{O}_X\}$.

- $\therefore$ #f.d. irreps$(A) \leq \dim \mathsf{HP}_0(\mathcal{O}_X)$; fixed bound for all A !

End of Part I.

II. Computational results (6)

- From now on, let V be a finite-dimensional symplectic vector space of dimension $2n > 0$ and $G < \mathsf{Sp}(V)$ a finite subgroup.
- I will describe some results later that reduce the computation of $\mathsf{HP}_0(\mathcal{O}_V^G)$ to a finite one.
- In fact, it also applies to $\mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V) := \mathcal{O}_V / \{\mathcal{O}_V^G, \mathcal{O}_V\}$.
- Note: $\mathsf{HP}_0(\mathcal{O}_V^G) = \mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)^G$.
- But first, I will give some computational results that use this and computer programs.

Coxeter groups of low rank (7)
Let $A = \mathsf{Weyl}(\mathbf{C}_{2n})$, quantizing $\mathcal{O}_{\mathbf{C}_{2n}}$.

Theorem 5. *For every Coxeter group $G < \mathsf{GL}_n < \mathsf{Sp}_{2n}$ of rank $n \leq 3$, $\mathsf{HP}_0(\mathcal{O}_{\mathbf{C}^{2n}}^G, \mathcal{O}_{\mathbf{C}_{2n}}) \cong \operatorname{gr} \mathsf{HH}_0(\mathsf{Weyl}(\mathbf{C}^{2n})^G, \mathsf{Weyl}(\mathbf{C}^{2n}))$. This is also true for $A_4, B_4 = C_4$, and D_4.*

- For G-invariants ($\mathsf{HP}_0(\mathcal{O}_V^G) \cong \mathsf{HH}_0(A^G)$) in the case of Weyl groups of rank ≤ 2 this is due to Alev and Foissy; for Weyl groups of rank ≤ 3 it is due to Butin.
- We compute in all of the above cases also the *grading* on HP_0 and hence $\operatorname{gr} \mathsf{HH}_0$, as well as the G-structure.
- In other joint work, I computed $h(\mathsf{HP}_0(\mathcal{O}_V^G); t)$ for types $B = C$ (more generally $H^n \rtimes S_n$ for $H < \mathsf{SL}_2(\mathbf{C})$), and in preparation are types A and D. The isomorphism $\mathsf{HP}_0(\mathcal{O}_V^G) \cong \mathsf{HH}_0(A^G)$ holds in types A, B, C but *not* for $D_n, n \geq 7$.

Complex reflection groups of rank two (8)

- Note that $\mathsf{GL}_n \hookrightarrow \mathsf{Sp}_{2n}$ via $M \mapsto \begin{pmatrix} M & 0 \\ 0 & (M^t)^{-1} \end{pmatrix}$.
- Let $G < \mathsf{GL}_n$ be a finite complex reflection group (i.e., generated by elements g such that $\operatorname{rank}(g - \mathrm{Id}) = 1$).
- These were classified by Shephard and Todd, and fall into the infinite families S_{n+1} and $G(m, p, n)$ (rank n, i.e., in GL_n), along with 34 exceptional groups, $G_4, G_5, \ldots, G_{37}$. The ones $G_4, \ldots, G_{22}$ have rank two (are in GL_2), along with $G(m, p, 2)$.

Theorem 6. *The complex reflection groups of rank two such that $\mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V) \cong \operatorname{gr} \mathsf{HH}_0(A^G, A)$ are exactly $S_3, G(m,1,2), G(m,m,2), G_4, G_6, G_8$, and G_{14}. The additional groups such that $\mathsf{HP}_0(\mathcal{O}_V^G) \cong \operatorname{gr} \mathsf{HH}_0(A^G)$ are $G(4,2,2), G(6,2,2), G_5, G_9$, and G_{21}.*

In all these cases, we explicitly computed the graded vector spaces above (with the G-structure), except G_{18}, G_{19}. **End of Part II.**

III. Finiteness theorem (9)

Theorem 7 (Berest, Etingof, Ginzburg '06). $\mathsf{HP}_0(\mathcal{O}_V^G)$ *is finite-dimensional.*

Same argument shows more generally: $\mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)$ is finite-dimensional.

Remark: In related joint work, I showed that $\mathsf{HP}_0(\mathcal{O}_X, \mathcal{O}_Y)$ is finite-dimensional when X has finitely many symplectic leaves and $X \to Y$ is a finite morphism (X is an affine Poisson variety and Y an affine variety). The proof used D-modules.

Proof of theorem (10)

- We show $\mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)^* \subseteq \mathcal{O}_V^*$ is finite-dimensional.
- Write $\mathcal{O}_V^* \cong \mathcal{O}_{V^*}$.
- For all $h \in \mathcal{O}_V^G$, let $\xi_h : \mathcal{O}_V \to \mathcal{O}_V$ be $\xi_h(f) := \{h, f\}$. So $\mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)^* = \{\phi \in \mathcal{O}_{V^*} \mid \xi_h^* \phi = 0, \forall h \in \mathcal{O}_V^G.\}$
- Explicitly, $\xi_h^* : \mathcal{O}_{V^*} \to \mathcal{O}_{V^*}$ is a differential operator, the Fourier transform of ξ_h: namely, $\xi_h^* = F(\xi_h)$, $F : \mathcal{D}_V \xrightarrow{\sim} \mathcal{D}_{V^*}^{\mathrm{op}}$ is the algebra anti-isomorphism sending multiplication by a linear function $u \in V^*$ to $\partial_u \in \mathcal{D}_{V^*}$, and differentiation $\partial_v, v \in V$ to multiplication by v.

Continuation of proof (11)

- Now, fix $u \in V^*$. For $\eta \in \mathcal{D}_{V^*}$, let $\eta|_u$ be obtained by evaluating the coefficients of η at u: i.e., $\eta|_u \in \mathsf{Sym}\, V^* = \mathbf{C}[\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}]$ for a basis $x_1, \ldots, x_n, y_1, \ldots, y_n$ of V^*.
- Then for $\phi \in \mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)^*$, $(\xi_h^*|_u \phi)(u) = (\xi_h^* \phi)(u) = 0$ for all $h \in \mathcal{O}_V^G$.
- Hence, the Taylor coefficients of ϕ at u are determined by $(\eta\phi)(u)$ where $\eta \in R_u := \mathsf{Sym}\, V^* / (\xi_h^*|_u : h \in \mathcal{O}_V^G)$.
- Thus $\dim \mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)^* \leq \dim R_u$.
- Claim: For generic $u \in V^*$, R_u is finite-dimensional. We will prove this in the next slide. $\square$

Remark: In fact, $\dim R_u$ is finite, and minimal, if u is not contained in $(V^K)^\perp$ for all nontrivial subgroups $K < G$. **End of part III.**

IV. A lemma on regular sequences (12)

Let U be an arbitrary finite-dimensional vector space, and set $m = \dim U$. Recall that a regular sequence $h_1, \ldots, h_k \in \mathcal{O}_U$ is one such that h_i is not a zero divisor in $\mathcal{O}_U/(h_1, \ldots, h_{i-1})$, i.e., the zero locus of $h_1, \ldots, h_k$ has codimension k.

Lemma 8. *If $h_1, \ldots, h_m \in \mathcal{O}_U$ is a regular sequence of homogeneous elements of degree ≥ 2, then for generic $u \in U$, the directional derivatives $D_u h_1, \ldots, D_u h_m$ also form a regular sequence.*

Proof:

- $\mathcal{O}_U$ is a finite module over the subalgebra $\mathbf{C}[h_1, \ldots, h_m]$, i.e., the inclusion defines a finite morphism $\phi : U \to \mathbf{A}^m$.
- Identify $TU \cong U \times U$ with the vertical component second.
- Let $Z := \{(v, u) \in U \times U : D_u\phi(v) = 0\}$, where $D_u\phi$ is the directional derivative in the direction u. (Really $(v, u) \in TU$).
- Let Y_u be the zero locus of $D_u h_1, \ldots, D_u h_m$. Then $(U \times \{u\}) \cap Z = Y_u \times \{u\}$.
- We need: $Y_u = \{0\}$ for generic u.

Completion of proof (13)

- Equivalently, we need $\dim Y_u = 0$ generically since Y_u is conical (cut by homogeneous equations of degree ≥ 1).
- Claim: $\dim Z \leq m$.
- By the claim, $(U \times \{u\}) \cap Z = Y_u \times \{u\}$ has dimension zero for generic u, as desired.
- Proof of claim: Let U_r be the locus where $\phi : U \to \mathbf{A}^m$ has rank r. Then $Z|_{U_r}$ is a vector bundle of rank $m - r$.
- It suffices to prove that $\dim U_r \leq r$.
- Since ϕ is finite, generically $\operatorname{rank}(\phi|_{U_r}) = \dim U_r$.
- By definition, $\phi|_{U_r}$ has rank $\leq r$. $\square$

Bound on $\dim R_u$ (14)

- Recall that $R_u = \mathsf{Sym}\, V^*/(\xi_h^*|_u : h \in \mathcal{O}_V^G)$.
- Let $\dim V = 2n$ and let $g_1, \ldots, g_{2n} \in \mathcal{O}_V^G$ be a regular sequence of homogeneous positively-graded elements.

- Note: $\xi_h^*|_u = F(\{h,u\})$, where F was the Fourier transform, $w \mapsto \partial_w$.
- So $\xi_h^*|_u = D_{u'}h$ where $u' \in V$ corresponds to $u \in V^*$ under the symplectic form on V.
- For generic u, the lemma implies that $(\xi_{g_1}^*|_u, \ldots, \xi_{g_{2n}}^*|_u) = (F(D_{u'}g_1), \ldots, F(D_{u'}g_{2n}))$ form a regular sequence.
- Set $R := \mathcal{O}_V/(D_{u'}g_i)$. Then $R \twoheadrightarrow R_u$, and $\dim R = \prod_{i=1}^{2n}(|D_{u'}g_i|) = \prod_{i=1}^{2n}(|g_i|-1)$.
- So $\dim \mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)^* \leq \dim R_u \leq \prod_{i=1}^{2n}(|g_i|-1)$.

Complex reflection groups (15)

- Let $G < \mathsf{GL}_n < \mathsf{Sp}_{2n}$ be a complex reflection group.
- Then by the Chevalley-Shephard-Todd theorem, $\mathbb{C}[x_1, \ldots, x_n]^G = \mathbb{C}[g_1, \ldots, g_n]$ is a polynomial algebra, with homogeneous g_i.
- Let $g_{n+1}, \ldots, g_{2n}$ be the corresponding elements in $\mathbb{C}[y_1, \ldots, y_n]^G$.
- Then $g_1, \ldots, g_{2n} \in \mathcal{O}_V^G$ is a regular sequence. We deduce:
- $\dim \mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V) \leq \prod(|g_i|-1) < \prod(|g_i|) = |G|^2$.

Problem with all this: computing HP_0 is still not a finite computation. We need a bound on the degree, not the dimension. **End of part IV.**

V. A bound on degree for $G < \mathsf{GL}_n < \mathsf{Sp}_{2n}$: Hilbert series bound (16)

- For $G < \mathsf{GL}_n$, write $V = U \oplus W$, $\mathcal{O}_U = \mathbf{C}[x_1, \ldots, x_n]$ and $\mathcal{O}_W = \mathbf{C}[y_1, \ldots, y_n]$, with U and W Lagrangian, and $(x_i, y_j) = \delta_{ij}$.
- Now rather than take $u \in V^*$ generic, we can take $u \in U^*$.
- The advantage: Taylor series coefficients of $f \in \mathcal{O}_{V^*}$ at $u \in U^*$ give the degree in W!
- Hence, the Taylor coefficients of $f \in \mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)^*$ of degree $-m$ in W are determined by differentiation by elements of R_u of degree m in W^*.
- Thus, $h(\mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)^*, \deg_W; t^{-1}) \leq h(R_u, \deg_{W^*}; t)$.
- Also, note that $\xi_{\sum_i x_i y_i}$ is the Euler vector field for $\deg_{U^*} - \deg_{W^*}$. So $|f| = 2\deg_{W^*} f$ for $f \in \mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)$.
- Hence $h(\mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V); t) \leq h(R_u, \deg_{W^*}; t^2)$.

Top degree bound (17)

- Problem: $u \in U^* \subset V^*$, so it can never be generic. The regular sequence lemma does not apply.
- Solution: Consider the image of R_u modulo the ideal (U^*) in $\mathbf{C}[W]$. The degree in W^* is unchanged.
- Let $\bar{R}_u := R_u/(U^*)$. Then $\bar{R}_u = \mathbf{C}[W]/(D_{u'}\bar{g}_i)$ where $\bar{g}_i$ are the images of g_i in $\mathbf{C}[W]$. (Note: $u' \in W$).
- Hence, the top degree of R_u in W^* equals that of $\bar{R}_u$.
- If $g_1, \ldots, g_n$ is a regular sequence in $\mathbf{C}[W]$, we conclude, for generic $u \in U^*$:

$$\operatorname{topdeg}(\mathsf{HP}_0(cO_V^G, \mathcal{O}_V)) \leq 2 \operatorname{topdeg}_{W^*}(\overline{R_u}) \leq 2 \sum_i (|g_i| - 2).$$

Complex reflection groups (18)

Corollary 9. *The top degrees of* $\mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)$ *for complex reflection groups* G *are at most:*

S_{n+1}:	$n(n-1)$	$G(m,1,1)$:	$2(m-2)$

$G(m,p,n)$, $m,n > 1$:	$n(n-1)m + 2mn/p - 4n$

G_4:	12	G_5:	28	G_6:	24	G_7:	40	G_8:	32
G_9:	56	G_{10}:	64	G_{11}:	88	G_{12} :	20	G_{13}:	32
G_{14}:	52	G_{15}:	64	G_{16}	92	G_{17}:	152	G_{18}:	172
G_{19}:	232	G_{20}:	76	G_{21}:	136	G_{22}:	56	G_{23}:	24
G_{24}:	36	G_{25}:	42	G_{26}:	60	G_{27}:	84	G_{28}:	40
G_{29}:	72	G_{30}:	112	G_{31}:	112	G_{32}:	152	G_{33}:	80

G_{34}:	240	G_{35}:	60	G_{36}:	112	G_{37}:	224

Problems: 1) this is still too high (above the actual top degree); 2) We cannot apply any of the above to groups not in $\mathsf{GL}_n < \mathsf{Sp}_{2n}$. **End of Part V.**

VI. A general bound on the Hilbert series (19)

- To ameliorate this, we construct a local system on the line $\mathbf{C} \cdot u$, so that we retain the (correct) notion of degree on the line:
- Let $f_1, \ldots, f_N$ be a basis for R_u (arbitrary $G < \mathsf{Sp}(V)$ and $u \in V^*$).
- Lift them to $F_1, \ldots, F_N \in \mathcal{D}_{V^*}$ (constant-coefficient diff. ops).
- Claim: For every $\phi \in \mathcal{D}_{V^*}$, there exists $\psi = \sum_i \lambda_i F_i$, $\lambda_i \in \mathbf{C}$ such that, for all $f \in \mathsf{HP}_0(\mathcal{O}_V, \mathcal{O}_V^G)^*$, $\phi(f)(u) = \psi(f)(u)$.
- The claim follows because ξ_h^* annihilate f.

The bound on $h(\mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V); t)$ continued (20)

- Recall: For every $\phi \in \mathcal{D}_{V^*}$, there exists $\psi = \sum_i \lambda_i F_i$, $\lambda_i \in \mathbf{C}$ such that, for all $f \in \mathsf{HP}_0(\mathcal{O}_V, \mathcal{O}_V^G)^*$, $\phi(f)(u) = \psi(f)(u)$.
- Using this claim: let $\xi \in \mathcal{D}_{V^*}$ be the Euler vector field: $[\xi, \eta] = |\eta|\eta$; $|v| = 1$ and $|\partial_w| = -1$ for $v \in V$ and $w \in V^*$.
- Let $C_\xi \in \mathrm{Mat}_N(\mathbf{C})$ be such that $(\xi F_1, \ldots, \xi F_n)(f)(u) = \big(C_\xi(F_1, \ldots, F_n)\big)(f)(u)$, $\forall f \in \mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)^*$.
- Then $(F_1(f)|_{\mathbf{C}\cdot u}, \ldots, F_N(f)|_{\mathbf{C}\cdot u})$ is an eigenvector of $B_\xi := C_\xi - \mathrm{Diag}(|F_1|, \ldots, |F_n|)$ of eigenvalue $|f|$.
- Conclusion: $h(\mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)^*; t) \leq \sum_i \dim E_i(B_\xi)$, where E_i is the eigenspace of eigenvalue i.
- This very often gives exactly the correct top degree!
- Can combine this with the bound in the GL_n case: using the GL_n bound we can compute the above matrix modulo p, for p greater than the bound given by the GL_n argument.

End of talk.