DEFORMATIONS OF ALGEBRAS IN NONCOMMUTATIVE ALGEBRAIC GEOMETRY

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Abstract. These are expanded lecture notes for the author's minicourse at MSRI in June 2012. In these notes, following, e.g., [Eti05, DTT09], we give an example-motivated review of the deformation theory of associative algebras in terms of the Hochschild cochain complex as well as quantization of Poisson structures, and Kontsevich's formality theorem in the smooth setting. We then discuss quantization and deformation via Calabi-Yau algebras and potentials. Examples discussed include Weyl algebras, enveloping algebras of Lie algebras, symplectic reflection algebras, quasihomogeneous isolated hypersurface singularities (including du Val singularities), and Calabi-Yau algebras.

Contents

We will work throughout over a field **k**. A lot of the time we will need it to have characteristic zero; feel free to simply assume this always (and we might forget to mention this assumption sometimes, even if it is necessary).

Guide to the reader: This text has a lot of remarks, not all of which are essential; so it is better to glance at them and decide whether they interest you rather than reading them all thoroughly.

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1. Lecture 1: Motivating examples

In this section, we will often work with graded algebras, and we always mean that algebras are equipped with an additional weight grading: so we are *not* working with superalgebras; all algebras in this section have homological degree zero and hence even parity. This will change in Lecture 2 (but won't really become important until Lecture 3).

1.1. Universal enveloping algebras of Lie algebras.

The first motivating example is that the enveloping algebra $U\mathfrak{g}$ deforms the symmetric algebra $\mathsf{Sym}\,\mathfrak{g}$. I expect this will be familiar to most students, but I give a brief review.

1.1.1. *The enveloping algebra.* Let $\mathfrak{g}$ be a Lie algebra with Lie bracket $\{-,-\}$. Then the representation theory of $\mathfrak{g}$ can be restated in terms of the representation theory of its *enveloping algebra*,

$$U\mathfrak{g} := T\mathfrak{g}/(xy - yx - \{x,y\} \mid x, y \in \mathfrak{g}), \tag{1.1}$$

where $T\mathfrak{g}$ is the tensor algebra of $\mathfrak{g}$ and $(-)$ is denotes the two-sided ideal generated by this relation.

Proposition 1.2. A representation V of $\mathfrak{g}$ is the same as a representation of $U\mathfrak{g}$.

Proof. If V is a representation of $\mathfrak{g}$, we define the action of $U\mathfrak{g}$ by

$$x_1 \cdots x_n(v) = x_1(x_2(\cdots(x_n(v))\cdots)).$$

We only have to check the relation defining $U\mathfrak{g}$:

$$(xy - yx - \{x,y\})(v) = x(y(v)) - y(x(v)) - \{x,y\}(v), \tag{1.3}$$

which is zero since the action of $\mathfrak{g}$ was a Lie action.

For the opposite direction, if V is a representation of $U\mathfrak{g}$, we define the action of $\mathfrak{g}$ by restriction: $x(v) = x(v)$. This defines a Lie action since the LHS of (1.3) is zero. □

Remark 1.4. More conceptually, the assignment $\mathfrak{g} \mapsto U\mathfrak{g}$ defines a functor from Lie algebras to associative algebras. Then the above statement says

$$\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{End}(V)) = \mathrm{Hom}_{\mathrm{Ass}}(U\mathfrak{g}, \mathrm{End}(V)),$$

where Lie denotes Lie algebra morphisms, and Ass denotes associative algebra morphisms. This statement is a consequence of the statement that $\mathfrak{g} \mapsto U\mathfrak{g}$ is a functor from Lie algebras to associative algebras which is left adjoint to the restriction functor $A \mapsto A^- := (A, \{a,b\} := ab - ba)$. Namely, we can write the above equivalently as

$$\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{End}(V)^-) = \mathrm{Hom}_{\mathrm{Ass}}(U\mathfrak{g}, \mathrm{End}(V)).$$

1.1.2. *$U\mathfrak{g}$ as a filtered deformation of* $\mathsf{Sym}\,\mathfrak{g}$. Next, we can define the commutative algebra $\mathsf{Sym}\,\mathfrak{g}$, the symmetric algebra of $\mathfrak{g}$. Concretely, if $\mathfrak{g}$ has basis $x_1, \ldots, x_n$, then $\mathsf{Sym}\,\mathfrak{g} = \mathbf{k}[x_1, \ldots, x_n]$ is the polynomial algebra in $x_1, \ldots, x_n$. More generally, $\mathsf{Sym}\,\mathfrak{g}$ can be defined by

$$\mathsf{Sym}\,\mathfrak{g} = T\mathfrak{g}/(xy - yx \mid x, y \in \mathfrak{g}).$$

The algebra $\mathsf{Sym}\,\mathfrak{g}$ is naturally graded by assigning $\mathfrak{g}$ degree one, i.e., a word $x_1 \cdots x_m$ with $x_i \in \mathfrak{g}$ is assigned degree m.

Similarly, $U\mathfrak{g}$ naturally has an increasing filtration by assigning $\mathfrak{g}$ degree one: that is, we define

$$(U\mathfrak{g})_{\leq m} = \langle x_1 \cdots x_j \mid x_1, \ldots, x_j \in \mathfrak{g}, j \leq m \rangle.$$

Definition 1.5. A increasing filtration on an associative algebra A is an assignment, to every $m \in \mathbf{Z}$, of a subspace $A_{\leq m} \subseteq A$, such that

$$A_{\leq m} \subseteq A_{\leq (m+1)}, \quad \forall m \in \mathbf{Z},$$
$$A_{\leq m} \cdot A_{\leq n} \subseteq A_{\leq m+n}, \forall m, n \in \mathbf{Z}.$$

This filtration is called nonnegative if $A_{\leq m} = 0$ whenever $m < 0$. It is called exhaustive if $A = \bigcup_{m \geq 0} A_m$.

We will only consider exhaustive filtrations, so we omit that term.

Definition 1.6. For any nonnegatively increasingly filtered algebra $A = \bigcup_{m \geq 0} A_m$, let $\mathsf{gr}\, A := \bigoplus_{m \geq 0} A_m / A_{m-1}$ be its associated graded algebra. Let $\mathsf{gr}_m A = (\mathsf{gr}\, A)_m = A_m / A_{m-1}$.

There is a surjection

$$\mathsf{Sym}\,\mathfrak{g} \twoheadrightarrow \mathsf{gr}\, U\mathfrak{g}, (x_1 \cdots x_m) \mapsto \mathsf{gr}(x_1 \cdots x_m), \tag{1.7}$$

which is well-defined since, in $\mathsf{gr}\, U\mathfrak{g}$, one has $xy - yx = 0$. The Poincaré-Birkhoff-Witt theorem states that this is an isomorphism:

Theorem 1.8. (PBW) The map (1.7) is an isomorphism.

The PBW is the *key* property that says that the deformations have been deformed in a *flat* way, so that the algebra has not gotten any smaller (the algebra cannot get bigger by deforming the relations, only smaller). This is a very special property of the deformed relations: see the following exercise.

Exercise 1.9. Suppose more generally that $B = TV/(R)$ for $R \subseteq TV$ a homogeneous subspace (i.e., R is spanned by homogeneous elements). Suppose also that $E \subseteq TV$ is an arbitrary filtered deformation of the relations, i.e., $\mathsf{gr}\, E = R$. Show that there is a canonical surjection

$$B \twoheadrightarrow \mathsf{gr}\, A.$$

So by deforming relations, the algebra A can only get smaller than B, and cannot get larger (and in general, it does get smaller—see the exercises!)

1.2. **Quantization of Poisson algebras.** As we will explain, the isomorphism (1.7) is compatible with the natural Poisson structure on $\mathsf{Sym}\,\mathfrak{g}$.

1.2.1. *Poisson algebras.*

Definition 1.10. A Poisson algebra is a commutative algebra B equipped with a Lie bracket $\{-,-\}$ satisfying the Leibniz identity:

$$\{ab, c\} = a\{b, c\} + b\{a, c\}.$$

Now, let $B := \mathsf{Sym}\,\mathfrak{g}$ for $\mathfrak{g}$ a Lie algebra. Then, B has a canonical Poisson bracket which extends the Lie bracket:

$$\{a_1 \cdots a_m, b_1 \cdots b_n\} = \sum_{i,j} \{a_i, b_j\} a_1 \cdots \hat{a}_i \cdots a_m b_1 \cdots \hat{b}_j \cdots b_n. \tag{1.11}$$

1.2.2. *Poisson structures on associated graded algebras.* Generally, let A be an increasingly filtered algebra such that $\mathsf{gr}\, A$ is commutative. Moreover, fix $d \geq 1$ such that

$$[A_{\leq m}, A_{\leq n}] \subseteq A_{\leq m+n-d}, \forall m, n. \tag{1.12}$$

One can always take $d = 1$, but in general we want to take d maximal so that the above property is satisfied.

We claim that, $\mathsf{gr}\, A$ is canonically Poisson, with the bracket, for $a \in A_{\leq m}$ and $b \in A_{\leq n}$,

$$\{\mathsf{gr}_m a, \mathsf{gr}_n b\} := \mathsf{gr}_{m+n-d}(ab - ba).$$

Note that $ab - ba$ is in $A_{\leq m+n-1}$ since $\mathsf{gr}\, A$ is commutative.

Exercise 1.13. Verify that the above is indeed a Poisson bracket, i.e., it satisfies the Lie and Leibniz identities.

We conclude that $\mathsf{Sym}\,\mathfrak{g}$ is equipped with a Poisson bracket by Theorem 1.8, i.e., by the isomorphism (1.7), taking $d := 1$.

Exercise 1.14. Verify that the Poisson bracket on $\mathsf{Sym}\,\mathfrak{g}$ obtained from (1.7) is the same as the one of (1.11), for $d := 1$.

1.2.3. *Filtered quantizations.* The preceding example motivates the definition of a filtered quantization:

Definition 1.15. Let B be a graded Poisson algebra, such that the Poisson bracket has negative degree $-d$. Then a *filtered quantization* of B is a filtered associative algebra A such that (1.12) is satisfied, such that $\mathsf{gr}\,A \cong B$ as Poisson algebras.

Again, the key PBW property here is that $\mathsf{gr}\,A \cong B$.

1.3. **Quantization of the nilpotent cone.**

1.3.1. *Central reductions.* Suppose, generally, that A is a filtered quantization of $B = \mathsf{gr}\,A$. Suppose in addition that there is a central filtered subalgebra $Z \subseteq A$. Then $\mathsf{gr}\,Z$ is Poisson central in B. For every character $\eta : Z \to \mathbf{k}$, we obtain central reductions

$$A_\eta := A/\ker(\eta)A, \quad B_\eta := /\,\mathsf{gr}(\ker\eta)B.$$

In the case that $B_0 = \mathbf{k}$ (or more generally $(\mathsf{gr}\,Z)_0 = \mathbf{k}$), which will be the case for us, note that B_η does not actually depend on η, and we obtain $B_\eta = B/(\mathsf{gr}\,Z)_+ B$, where $(\mathsf{gr}\,Z)_+ \subseteq \mathsf{gr}\,Z$ is the augmentation ideal (the ideal of positively-graded elements).

Now let us restrict to our situation of $A = U\mathfrak{g}$ and $B = \mathsf{Sym}\,\mathfrak{g}$ with $\mathbf{k}$ of characteristic zero. Let us suppose moreover that $\mathfrak{g}$ is finite-dimensional semisimple. Then, the center $Z(A)$ is well known. The central reduction $(U\mathfrak{g})_\eta$ is the algebra whose representations are those $\mathfrak{g}$-representations with central character η, i.e., representations V such that, for all $v \in V$ and $z \in Z(A)$, $z \cdot v = \eta(z) \cdot v$. (We remark that the subcategories $\mathrm{Rep}(A_\eta) \subseteq \mathrm{Rep}(A)$ are all orthogonal, in that there are no nontrivial homomorphisms or extensions between representations with distinct central characters.)

Example 1.16. Let $\mathfrak{g} = \mathfrak{sl}_2 = \langle e, f, h\rangle$. Then $Z(U\mathfrak{g}) = \mathbf{k}[C]$, where the element C is the Casimir element, $C = ef + fe + \frac{1}{2}h^2$. In this case, the central reduction $(U\mathfrak{g})_\eta$ describes those representations on which C acts by a fixed scalar $\eta(C)$. For example, there exists a finite-dimensional representation of $(U\mathfrak{g})_\eta$ if and only if $\eta(C) \in \{m + \frac{1}{2}m^2 \mid m \geq 0\}$, since $\eta(C)$ acts on a highest-weight vector v of h of weight λ, i.e., a vector such that $ev = 0$ and $hv = \lambda v$, by $C \cdot v = (h + \frac{1}{2}h^2)v = (\lambda + \frac{1}{2}\lambda^2)v$. In particular, there are only countably many such characters η that admit a finite-dimensional representation.

Moreover, when $\eta(C) \in \{m + \frac{1}{2}m^2 \mid m \geq 0\}$, there is *exactly one* finite-dimensional representation of $(U\mathfrak{g})_\eta$: the one with highest weight m. So these quantizations $(U\mathfrak{g})_\eta$ of $(\mathsf{Sym}\,\mathfrak{g})/(\mathsf{gr}\,C)$ have at most one finite-dimensional representation, and only countably many have this finite-dimensional representation.

More generally, if $\mathfrak{g}$ is finite-dimensional semisimple (still with $\mathbf{k}$ of characteristic zero), it turns out that $Z(U\mathfrak{g})$ is a polynomial algebra, and $\mathsf{gr}\,Z(U\mathfrak{g}) \to Z(\mathsf{Sym}\,\mathfrak{g})$ is an isomorphism of polynomial algebras:

Theorem 1.17. Let $\mathfrak{g}$ be finite-dimensional semisimple and $\mathbf{k}$ of characteristic zero. Then $Z(U\mathfrak{g}) \cong \mathbf{k}[x_1, \ldots, x_r]$ is a polynomial algebra, with r equal to the semisimple rank of $\mathfrak{g}$. Moreover, the polynomial algebra $\mathsf{gr}\,Z(U\mathfrak{g})$ equals the Poisson center $Z(\mathsf{Sym}\,\mathfrak{g}) = (\mathsf{Sym}\,\mathfrak{g})^{\mathfrak{g}}$ of $\mathsf{gr}\,U\mathfrak{g}$.

We remark that the isomorphism $\mathsf{gr}\,Z(U\mathfrak{g}) \xrightarrow{\sim} Z(\mathsf{gr}\,U\mathfrak{g}) = Z(\mathsf{Sym}\,\mathfrak{g})$ actually holds for *arbitrary* finite-dimensional $\mathfrak{g}$, by the Kirillov-Duflo theorem; we will explain how this follows from Kontsevich's formality theorem in Corollary 4.52.

The degrees d_i of the generators $\mathsf{gr}\, x_i$ are known as the *fundamental degrees*, and they satisfy $d_i = m_i + 1$ where m_i are the Coxeter exponents of the associated root system, cf. e.g., [Hum90]).

By the theorem, for every character $\eta : Z(U\mathfrak{g}) \to \mathbf{k}$, one obtains an algebra $(U\mathfrak{g})_\eta$ which quantizes $(\mathsf{Sym}\,\mathfrak{g})/((\mathsf{Sym}\,\mathfrak{g})^{\mathfrak{g}}_+)$. Here $(\mathsf{Sym}\,\mathfrak{g})^{\mathfrak{g}}_+$ is the augmentation ideal of $\mathsf{Sym}\,\mathfrak{g}$, which equals $\mathsf{gr}(\ker\eta)$ since $(\mathsf{Sym}\,\mathfrak{g})_0 = \mathbf{k}$ (cf. the comments above).

Definition 1.18. The *nilpotent cone* $\mathrm{Nil}\,\mathfrak{g} \subseteq \mathfrak{g}$ is the set of all nilpotent elements of $\mathfrak{g}$, i.e., elements $x \in \mathfrak{g}$ such that, for some $N \geq 1$, $(\mathrm{ad}\, x)^N = 0$, i.e., $[x,[x,\cdots,[x,y]]] = 0$ for all $x \in \mathfrak{g}$ where x appears N times in the iterated commutators.

It is clear that $\mathrm{Nil}\,\mathfrak{g}$ is a cone, i.e., if $x \in \mathrm{Nil}\,\mathfrak{g}$, then so is $c \cdot x$ for all $c \in \mathbf{k}$. In other words, $\mathrm{Nil}\,\mathfrak{g}$ is conical: it has a contracting $\mathbf{G}_m$ action with fixed point 0.

Proposition 1.19. Let $\mathfrak{g}$ be finite-dimensional semisimple, so $\mathfrak{g} \cong \mathfrak{g}^*$. Then, the algebra B_0 is the algebra of functions on the nilpotent cone $\mathrm{Nil}\,\mathfrak{g} \subseteq \mathfrak{g}$.

Proof. Let G be a connected algebraic group such that $\mathrm{Lie}\, G = \mathfrak{g}$. Recall that $(\mathsf{Sym}\,\mathfrak{g})^{\mathfrak{g}} \cong \mathcal{O}_{\mathfrak{h}/W}$. This maps the augmentation ideal $(\mathsf{Sym}\,\mathfrak{g})^{\mathfrak{g}}_+$ to the augmentation ideal of $(\mathcal{O}_{\mathfrak{h}/W})_+$, which is the ideal of the zero element $0 \in \mathfrak{h}^* \cong \mathfrak{h}$. The ideal $(\mathcal{O}_{\mathfrak{h}/W})_+ \cdot \mathsf{Sym}\,\mathfrak{g}$ thus defines those elements $x \in \mathfrak{g}^* \cong \mathfrak{g}$ such that $\overline{G \cdot x} \cap \mathfrak{h} = \{0\}$. These are exactly the elements whose semisimple part is zero, i.e., the nilpotent elements. □

Corollary 1.20. For $\mathfrak{g}$ finite-dimensional semisimple, the central reductions $(U\mathfrak{g})_\eta := U\mathfrak{g}/\ker\eta$ quantize $\mathcal{O}_{\mathrm{Nil}\,\mathfrak{g}}$.

Example 1.21. In the case $\mathfrak{g} = \mathfrak{sl}_2$, the quantizations $(U\mathfrak{g})_\eta$, whose representations are those $\mathfrak{sl}_2$-representations on which the Casimir C acts by a fixed scalar, all quantize the cone of nilpotent 2×2 matrices,

$$\mathrm{Nil}(\mathfrak{sl}_2) = \left\langle \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a^2 + bc = 0 \right\rangle.$$

Remark 1.22. Theorem 1.17 is stated anachronistically and deserves some explanation. Originally it was known that $(\mathsf{Sym}\,\mathfrak{g})^{\mathfrak{g}} \cong (\mathsf{Sym}\,\mathfrak{h})^W$ where $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra and W is the Weyl group (since $(\mathsf{Sym}\,\mathfrak{g})^{\mathfrak{g}} = \mathcal{O}_{\mathfrak{g}^*//G}$, the functions on coadjoint orbits in $\mathfrak{g}^*$; these are identified with adjoint orbits of G in $\mathfrak{g}$ by the Killing form $\mathfrak{g}^* \cong \mathfrak{g}$; then the closed orbits all contain points of $\mathfrak{h}$, and their intersections with $\mathfrak{h}$ are exactly the W-orbits.) Then, Coxeter observed that $(\mathsf{Sym}\,\mathfrak{h})^W$ is a polynomial algebra, and the degrees d_i were computed by Shephard and Todd. After that, Harish-Chandra constructed an explicit isomorphism $HC : Z(U\mathfrak{g}) \xrightarrow{\sim} \mathsf{Sym}(\mathfrak{h})^W$, such that, for every highest-weight representation V_λ of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$ and (nonzero) highest weight vector $v \in V_\lambda$, one has

$$z \cdot v = HC(z)(\lambda) \cdot v,$$

viewing $HC(z)$ as a polynomial function on $\mathfrak{h}^*$. The Harish-Chandra isomorphism is nontrivial: indeed, the target of HC equips $\mathfrak{h}^*$ not with the usual action of W, but the *affine* action, defined by $w \cdot \lambda := w(\lambda + \delta) - \delta$, where the RHS uses the usual action of W on $\mathfrak{h}$, and δ is the sum of the fundamental weights. This shifting phenomenon is common when quantizing a center. In this case one can explicitly see why the shift occurs because the center must act by the same character on highest weight representations V_λ and $V_{w\cdot\lambda}$, since when λ is dominant, and these are the Verma modules, then $V_{w\cdot\lambda} \subseteq V_\lambda$.

1.4. **Weyl algebras.** The next important example is that of Weyl algebras. Let $\mathbf{k}$ have characteristic zero. Letting $\mathbf{k}\langle x_1, \ldots, x_n\rangle$ denote the noncommutative polynomial algebra in variables $x_1, \ldots, x_n$, this is typically defined by

$$A_n := \mathbf{k}\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle/([x_i, y_j] = \delta_{ij}, [x_i, x_j] = 0 = [y_i, y_j]),$$

denoting here $[a,b] := ab - ba$.

Exercise 1.23. Show that, setting $y_i := -\partial_i$, one obtains an isomorphism from A_n to the algebra, $\mathcal{D}_{\mathbf{A}^n}$, of differential operators on $\mathsf{Sym}[x_1, \ldots, x_n] = \mathcal{O}_{\mathbf{A}^n}$ with polynomial coefficients.

More invariantly:

Definition 1.24. Let V be a symplectic vector space with form $(-,-)$. Then, the Weyl algebra $\mathsf{Weyl}(V)$ is defined by

$$\mathsf{Weyl}(V) = TV/(xy - yx - (x,y)).$$

The Weyl algebra is equipped with two different filtrations. We first consider the *additive* or *Bernstein* filtration, given by $|x_i| = |y_i| = 1$, so then

$$\mathsf{Weyl}(V)_{\leq m} = \{v_1 \cdots v_m \mid v_i \in V\}.$$

Exercise 1.25. Equip $\mathsf{Sym}\, V$ with the unique Poisson bracket such that $\{v,w\} = (v,w)$ for $v,w \in V$. This bracket has degree -2. Show that, with the additive filtration, $\mathsf{gr}\,\mathsf{Weyl}(V) \cong \mathsf{Sym}\, V$ as Poisson algebras, where $d = 2$ in (1.12).

The *geometric* filtration assigns $|x_i| = 0$ and $|y_i| = 1$, which we will discuss in §1.5 below. For this choice of filtration, $d = 1$. This filtration has the advantage that it generalizes from $\mathsf{Weyl}(V)$ to the setting of differential operators on arbitrary varieties, but the disadvantage that the full symmetry group $\mathsf{Sp}(V)$ does not preserve the filtration, but only the subgroup $\mathsf{GL}(U)$, where $U = \mathrm{Span}\langle x_i \rangle$.

1.4.1. *Invariant subalgebras of Weyl algebras.* Note that $\mathsf{Sp}(V)$ acts by automorphisms of the Poisson algebra $\mathsf{Sym}\, V$ as well as the Weyl algebra $\mathsf{Weyl}(V)$. Let $G < \mathsf{Sp}(V)$ be a finite subgroup of order relatively prime to the characteristic of $\mathbf{k}$. Then, one can consider the invariant subalgebras $\mathsf{Weyl}(V)^G$ and $\mathsf{Sym}(V)^G$. One then easily sees that $\mathsf{Weyl}(V)^G$ is filtered (using the additive filtration!) and that

$$\mathsf{gr}(\mathsf{Weyl}(V)^G) \cong \mathsf{Sym}(V)^G = \mathcal{O}_{V^*/G}.$$

Moreover, as before, this is an isomorphism of Poisson algebras with $d = -2$.

Corollary 1.26. $\mathsf{Weyl}(V)^G$ is a filtered quantization of $\mathsf{Sym}(V)^G$.

On exercises: the algebra $\mathsf{Weyl}(V)^G$ can be further deformed, and this yields the so-called spherical symplectic reflection algebras, which are an important subject of Bellamy's lectures.

Example 1.27. The simplest case is already interesting: $V = \mathbf{k}^2$ and $G = \{\pm \mathrm{Id}\} \cong \mathbf{Z}/2$, and $\mathbf{k}$ does not have characteristic two. Then, $\mathsf{Sym}(V)^G = \mathbf{k}[x^2, xy, y^2] \cong \mathbf{k}[u,v,w]/(v^2 = uw)$, the algebra of functions on a singular quadric hypersurface in $\mathbf{A}^3$. The quantization, $\mathsf{Weyl}(V)^G$, on the other hand, is *homologically smooth*: it has finite global dimension and finite Hochschild dimension (equal to $\dim V$); it is in fact a *Calabi-Yau algebra of dimension* $\dim V$. So, quantizing the singular quadric yields something smooth, in this case.

Remark 1.28. Note, moreover, that this quadric $v^2 = uw$ is isomorphic to the quadric $a^2 + bc = 0$ of Example 1.16, i.e., $\mathbf{A}^2/(\mathbf{Z}/2) \cong \mathrm{Nil}(\mathfrak{sl}_2)$. Thus we have given *two quantizations* of the same variety: one by the invariant Weyl algebra, $\mathsf{Weyl}(\mathbf{k}^2)^{\mathbf{Z}/2}$, and the other a family of quantizations given by the central reductions $(U\mathfrak{sl}_2)_\eta = U\mathfrak{sl}_2/(C - \eta(C))$. In fact, the latter family is a universal family of quantizations, and one can see that $\mathsf{Weyl}(\mathbf{k}^2)^{\mathbf{Z}/2} \cong (U\mathfrak{sl}_2)_\eta$ where $\eta(C) = -\frac{3}{8}$ (see the exercises!).

This coincidence is the first case of a part of the *McKay correspondence*, which identifies, for every finite subgroup $G < \mathsf{SL}_2(\mathbf{C})$, the quotient $\mathbf{C}^2/G$ with a certain two-dimensional "Slodowy" slice of $\mathrm{Nil}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra whose Dynkin diagram has vertices labeled by the irreducible representations of G, and a single edge from V to W if and only if $V \otimes W$ contains a copy of the standard representation $\mathbf{C}^2$.

1.5. **Algebras of differential operators.** As mentioned above, the Weyl algebra $A_n = \mathsf{Weyl}(\mathbf{k}^{2n})$ is isomorphic to the algebra of differential operators on $\mathbf{k}^n$ with polynomial coefficients.

More generally, we can define

Definition 1.29 (Grothendieck)**.** Let B be a commutative $\mathbf{k}$-algebra. We define the space $\mathrm{Diff}_{\leq m}(B)$ of differential operators *of order* $\leq m$ inductively on m. For $a \in B$ and $\phi \in \mathrm{End}_{\mathbf{k}}(B)$, let $[\phi, a] \in \mathrm{End}_{\mathbf{k}}(B)$ be the linear operator

$$[\phi, a](b) := \phi(ab) - a\phi(b), \forall b \in B.$$

$$\mathrm{Diff}_{\leq 0}(B) = \{\phi \in \mathrm{End}_{\mathbf{k}}(B) \mid [\phi, a] = 0, \forall a \in B\} = \mathrm{End}_B(B) \cong B; \tag{1.30}$$

$$\mathrm{Diff}_{\leq m}(B) = \{\phi \in \mathrm{End}_{\mathbf{k}}(B) \mid [\phi, a] \in \mathrm{Diff}_{\leq (m-1)}(B), \forall a \in B\}. \tag{1.31}$$

Let $\mathrm{Diff}(B) := \bigcup_{m \geq 0} \mathrm{Diff}_{\leq m}(B)$.

Exercise 1.32. Verify that $\mathrm{Diff}(B)$ is a nonnegatively filtered associative algebra whose associated graded algebra is commutative.

Now, suppose that $B = \mathcal{O}_X$ is finitely-generated. Define the *tangent sheaf* T_X to be the B-module

$$T_X := \mathrm{Der}_{\mathbf{k}}(\mathcal{O}_X, \mathcal{O}_X),$$

of derivations of $\mathcal{O}_X$. This is naturally a $\mathcal{O}_X$-module, hence a coherent sheaf. A *global vector field* on X is a derivation $\xi \in \mathrm{Der}_{\mathbf{k}}(\mathcal{O}_X, \mathcal{O}_X)$, i.e., a global section of T_X.

Similarly, one can define the *cotangent sheaf*, as the $\mathcal{O}_X$-module of Kähler differentials

$$T^*_X := \langle a \cdot db, a, b \in \mathcal{O}_X \rangle / \langle d(ab) - a \cdot db - b \cdot da \rangle.$$

Then, $T_X = \mathrm{Hom}_{\mathcal{O}_X}(T^*_X, \mathcal{O}_X)$.

Now we can state the result that, for smooth (affine) varieties, $\mathrm{Diff}(\mathcal{O}_X)$ quantizes the cotangent bundle:

Proposition 1.33. If X is a smooth (affine) variety over $\mathbf{k}$ of characteristic zero, then as Poisson algebras,

$$\mathsf{gr}\,\mathrm{Diff}(\mathcal{O}_X) \cong \mathsf{Sym}_{\mathcal{O}_X} T_X \cong \mathcal{O}_{T^*X}.$$

We put "affine" in parentheses because the statement is valid with the same proofin the nonaffine context, provided all the objects are understood as sheaves on X.

The proposition requires some clarifications. First, since T_X is actually a Lie algebra, we obtain a Poisson structure on $\mathsf{Sym}_{\mathcal{O}_X} T_X$ by the formula

$$\{\xi_1 \cdots \xi_m, \eta_1 \cdots \eta_n\} = \sum_{i,j} [\xi_i, \eta_j] \cdot \xi_1 \cdots \hat{\xi}_i \cdots \xi_m \eta_1 \cdots \hat{\eta}_j \cdots \eta_n.$$

Above, the meaning of $\mathcal{O}_{T^*X}$ is given as follows: if M is a $\mathcal{O}_X$-module, i.e., a coherent sheaf on the affine variety X, then the total space of M is

$$\mathsf{Spec} \bigoplus_{m \geq 0} \mathrm{Hom}(\mathsf{Sym}^m_{\mathcal{O}_X} M, \mathcal{O}_X),$$

since the summand at each value of m is the collection of functions on the total space of M which have fiberwise degree m. More generally, if X is not affine, this construction produces a sheaf of algebras over $\mathcal{O}_X$ which defines $T^*X \to X$.

Proposition 1.33 says that, for smooth (affine) varieties, the algebra $\mathrm{Diff}(\mathcal{O}_X)$ quantizes $\mathcal{O}_{T^*X}$.

Remark 1.34. More conceptually, T_X is not merely a Lie algebra, but actually a Lie algebroid, since it acts by derivations on $\mathcal{O}_X$, and this action, the Lie structure, and the $\mathcal{O}_X$-module structure on T_X are all compatible by the axioms $[\xi, a] = \xi(a)$ for $\xi \in T_X$ and $a \in \mathcal{O}_X$. In this context, $\mathsf{Sym}_{\mathcal{O}_X} T_X$ is always a Poisson algebra.

Also, in this context, one can define the universal enveloping algebroid,

$$U_{\mathcal{O}_X} T_X := T_{\mathcal{O}_X}(T_X)/(\xi \cdot a - a \cdot \xi - \xi(a), \xi \cdot \eta - \eta \cdot \xi - [\xi, \eta]),$$

where $T_{\mathcal{O}_X} T_X$ denotes the tensor algebra of T_X over $\mathcal{O}_X$. Then, if T_X is free, or more generally locally free, then the PBW theorem generalizes to show that the natural map $\mathsf{Sym}_{\mathcal{O}_X} T_X \to \mathsf{gr}\, U_{\mathcal{O}_X} T_X$ is an isomorphism. In particular, this holds when X is smooth. Then, the proposition says that, in addition, the canonical map

$$U_{\mathcal{O}_X} T_X \to \mathrm{Diff}(\mathcal{O}_X) \tag{1.35}$$

is an isomorphism.

In these terms, we can explain the proof of the proposition, which we reduced to the fact that (1.35) is an isomorphism. The difficulty is surjectivity, which boils down to the statement that $\mathrm{Diff}(\mathcal{O}_X)$ is generated as an algebra by $\mathcal{O}_X$ and T_X. To prove it, it suffices to show that on associated graded, $\mathsf{Sym}_{\mathcal{O}_X} T_X \to \mathsf{gr}\, \mathrm{Diff}(\mathcal{O}_X)$ is surjective. This statement can be checked locally, in the formal neighborhood of each point $x \in X$, which is isomorphic to a formal neighborhood of affine space of the same dimension at the origin. But then one obtains $\mathbf{k}[\![x_1, \ldots, x_n]\!][\partial_1, \ldots, \partial_n]$, a completion of Weyl algebra, which is generated by constant vector fields and functions, proving the statement. (Alternatively, one can use ordinary localization at x; since $\dim \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim X$, Nakayama's lemma implies that a basis for this vector space lifts to a collection of $\dim X$ algebra generators of the local ring $\mathcal{O}_{X,x}$, and then differential operators of order $\leq m$ are determined by their action on products of $\leq m$ generators, and all possible actions are given by $\mathsf{Sym}^{\leq m}_{\mathcal{O}_{X,x}} T_{X,x}$.)

1.5.1. *Preview: $\mathcal{D}$-modules.* Since Proposition 1.33 does not apply to singular varieties, it is less clear how to treat algebras of differential operators and their modules in the singular case.

Later on, we will adopt the approach of right $\mathcal{D}$-modules. In this approach, one takes a singular variety X and embeds it into a smooth variety V, and defines the category of right $\mathcal{D}$-modules on X to be the category of right $\mathcal{D}_V$-modules supported on X, i.e., right modules M over the ring $\mathcal{D}_V$ of differential operators on V, with the property that, for all $m \in M$, there exists $N \geq 1$ such that $m \cdot I_X^N = 0$, where I_X is the ideal corresponding to X.

This circumvents the problem that the ring $\mathcal{D}_X$ of differential operators is not well-behaved, and one obtains a very useful theory, which is equivalent to two alternatives: the category of *!-crystals*, and the category of *perverse sheaves* (the latter should be thought of as gluings of local systems on subvarieties, or more precisely, complexes of such gluings with a particular property).

1.6. **Invariant differential operators.** It is clear that the group of automorphisms $\mathrm{Aut}(X)$ of the variety X acts by filtered automorphisms also on $\mathrm{Diff}(X)$ and also by graded automorphisms of $\mathsf{gr}\, \mathrm{Diff}(X) = \mathsf{Sym}_{\mathcal{O}_X} T_X$. Let us continue to assume $\mathbf{k}$ has characteristic zero.

Now, suppose that $G < \mathrm{Aut}(X)$ is a finite subgroup of automorphisms of X. Then one can form the algebras $\mathrm{Diff}(X)^G$ and $(\mathsf{Sym}_{\mathcal{O}_X} T_X)^G$. By Proposition 1.33, we conclude that $\mathsf{gr}\, \mathrm{Diff}(X)^G$ is a quantization of $(\mathsf{Sym}_{\mathcal{O}_X} T_X)^G$, i.e., it is a quantization.

One example of this is when $X = \mathbf{A}^n$ and $G < \mathsf{GL}(n) < \mathrm{Aut}(\mathbf{A}^n)$. Then we obtain that $\mathsf{Weyl}(\mathbf{A}^{2n})$ is a quantization of $\mathcal{O}_{T^*\mathbf{A}^n} = \mathcal{O}_{\mathbf{A}^{2n}}$, which is the special case of the example of §1.4.1 where $G < \mathsf{GL}(n)$ (note that $\mathsf{GL}(n) < \mathsf{Sp}(2n)$, where explicitly, a matrix A acts by the block matrix $\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$, with A^t denoting the transpose of A.)

Preview: By deforming $\mathrm{Diff}(X)^G$, one obtains global versions of the spherical rational Cherednik algebras, studied in [Eti04].

Example 1.36. Let $X = \mathbf{A}^1 \setminus \{0\}$ and let $G = \{1, g\}$ where $g(x) = x^{-1}$. Then $(T^*X)/G$ is a "global" or "multiplicative" version of the variety $\mathbf{A}^2/(\mathbf{Z}/2)$ of Example 1.27. The quantization is $\mathcal{D}_X^G$, and one has, by the above,

$$\mathsf{gr}\,\mathcal{D}_{\mathbf{A}^1\setminus\{0\}}^{\mathbf{Z}/2} \cong \mathcal{O}_{T^*(bA^1\setminus\{0\})}^{\mathbf{Z}/2}.$$

Explicitly, the action on tangent vectors is $g(\partial_x) = -x^2\partial_x$, so that $g(\partial_x)g(x) = 1 = -x^2\partial_x(x^{-1})$. Thus, setting $y := \mathsf{gr}\,\partial_x$, we have $g(y) = -x^2 y$. So $\mathcal{O}_{T^*X/G} = \mathbf{C}[x + x^{-1}, y - x^2 y, x^2 y^2]$ and $\mathcal{D}_X^G = \mathbf{C}[x + x^{-1}, \partial_x - x^2\partial_x, (x\partial_x)^2] \subseteq \mathcal{D}_X^G$.

1.7. **Beilinson-Bernstein localization theorem and global quantization.** The purpose of this section is to explain a deep property of the quantization of the nilpotent cone discussed above, namely that this *resolves* to a global (nonaffine) symplectic quantization, namely of a cotangent bundle. We still assume that $\mathrm{char}\,\mathbf{k} = 0$.

Return to the example $\mathrm{Nil}(\mathfrak{sl}_2) \cong \mathbf{A}^2/\{\pm \mathrm{Id}\}$. There is another way to view this Poisson algebra, by the *Springer resolution*. Namely, a nonzero nilpotent element $x \in \mathfrak{sl}_2$, up to scaling, is uniquely determined by the line $\ker(x) = \mathrm{im}(x)$ in $\mathbf{k}^2$. Consider the locus of pairs

$$X := \{(\ell, x) \in \mathbf{P}^1 \times \mathrm{Nil}(\mathfrak{sl}_2) \mid \ker(x) \subseteq \ell\} \subseteq \mathbf{P}^1 \times \mathrm{Nil}(\mathfrak{sl}_2).$$

This projects to $\mathrm{Nil}(\mathfrak{sl}_2)$. Moreover, the fiber over $x \neq 0$ is evidently a single point, since $\ker(x)$ determines ℓ. Only over the singular point $0 \in \mathrm{Nil}(\mathfrak{sl}_2)$ is there a larger fiber, namely $\mathbf{P}^1$ itself.

Lemma 1.37. *$X \cong T^*\mathbf{P}^1$.*

Proof. Fix $\ell \in \mathbf{P}^1$. Note that $T_\ell\mathbf{P}^1$ is naturally $\mathrm{Hom}(\ell, \mathbf{k}^2/\ell)$. On the other hand, the locus of x such that $\ker(x) \subseteq \ell$ naturally acts linearly on $T_\ell\mathbf{P}^1$: given such an x and given $\phi \in \mathrm{Hom}(\ell, \mathbf{k}^2/\ell)$, we can take $x \circ \phi \in \mathrm{Hom}(\ell, \ell) \cong \mathbf{k}$. Since this locus of x is a one-dimensional vector space, we deduce that it is $T_\ell^*\mathbf{P}^1$, as desired. ☐

Thus, we obtain a resolution of singularities

$$T^*\mathbf{P}^1 \twoheadrightarrow \mathrm{Nil}(\mathfrak{sl}_2) = \mathbf{A}^2/\{\pm \mathrm{Id}\}. \tag{1.38}$$

Note that $\mathrm{Nil}(\mathfrak{sl}_2)$ is affine, so that, since the morphism is birational (as all resolutions must be), $\mathcal{O}_{\mathrm{Nil}(\mathfrak{sl}_2)}$ is the algebra of global sections $\Gamma(T^*\mathbf{P}^1, \mathcal{O}_{T^*\mathbf{P}^1})$ of functions on $T^*\mathbf{P}^1$.

Moreover, equipping $T^*\mathbf{P}^1$ with its standard symplectic structure, (1.38) is a Poisson morphism: the Poisson structure on $\mathcal{O}_{\mathrm{Nil}(\mathfrak{sl}_2)}$ is obtained from the one on $\mathcal{O}_{T^*\mathbf{P}^1}$ by taking global sections.

In fact, this morphism can be *quantized*: let $\mathcal{D}_{\mathbf{P}^1}$ be the sheaf of differential operators with polynomial coefficients on $\mathbf{P}^1$. This quantizes $\mathcal{O}_{T^*\mathbf{P}^1}$, as we explained, since $\mathbf{P}^1$ is smooth (this fact works for nonaffine schemes as well, if one uses sheaves of algebras). Then, there is the deep

Theorem 1.39. (Beilinson-Bernstein for $\mathfrak{sl}_2$)

(i) There is an isomorphism of algebras $\Gamma(\mathbf{P}^1, \mathcal{D}_{\mathbf{P}^1}), (U\mathfrak{g})_{\eta_0}$, where $\eta_0(C) = 0$;

(ii) Taking global sections yields an equivalence of abelian categories

$$\mathcal{D}_{\mathbf{P}^1} - \mathrm{mod} \xrightarrow{\sim} (U\mathfrak{g})_{\eta_0} - \mathrm{mod}.$$

This quantizes the Springer resolution (1.38).

Remark 1.40. This implies that the quantization $\mathcal{D}_{\mathbf{P}^1}$ of $\mathcal{O}_{T^*\mathbf{P}^1}$ is, in a sense, **affine**, since its category of representations is equivalent to the category of representations of its global sections. For $\mathcal{O}_X$ for an arbitrary scheme X, this is true if and only if X is affine. But, even though $\mathbf{P}^1$ is projective (the opposite of affine), the noncommutative algebra $\mathcal{D}_{\mathbf{P}^1}$ is still affine, in a sense.

There is a longstanding conjecture that says that, if $\mathcal{D}_X$ is affine in the sense that $\mathcal{D}_X - \text{mod} \xrightarrow{\sim} \Gamma(X, \mathcal{D}_X) - \text{mod}$, then X is of the form $X = T^*(G/P)$ where $P < G$ is a parabolic subgroup. (The converse is a theorem of Beilinson-Bernstein, which generalizes Theorem 1.41 below to the parabolic case G/P instead of G/B.) Similarly, there is also an (even more famous) "associated graded" version, which says that if X is a smooth variety and $T^*X \twoheadrightarrow Y$ is a symplectic resolution with Y affine (i.e., $Y = \mathsf{Spec}\,\Gamma(T^*X, \mathcal{O}_{T^*X})$), then $X = G/P$.

In fact, this whole story generalizes to arbitrary connected semisimple algebraic groups G with $\mathfrak{g} := \operatorname{Lie} G$ the associated finite-dimensional semisimple Lie algebra. Let $\mathcal{B}$ denote the flag variety of G, which can be defined as the symmetric space G/B for $B < G$ a Borel subgroup.

Theorem 1.41. (i) (Springer resolution) There is a symplectic resolution $T^*\mathcal{B} \to \operatorname{Nil}(\mathfrak{g})$, which is the composition

$$T^*\mathcal{B} = \{(\mathfrak{b}, x)\} \subseteq (\mathcal{B} \times \mathfrak{g}) \twoheadrightarrow \operatorname{Nil}(\mathfrak{g}),$$

where the last map is the second projection;

(ii) (Beilinson-Bernstein) There is an isomorphism $\Gamma(\mathcal{B}, \mathcal{D}_{\mathcal{B}}) \cong (U\mathfrak{g})_{\eta_0}$, where $\ker(\eta_0)$ acts by zero on the trivial representation of $\mathfrak{g}$;

(iii) (Beilinson-Bernstein) Taking global sections yields an equivalence of abelian categories, $\mathcal{D}_{\mathcal{B}} - \text{mod} \xrightarrow{\sim} (U\mathfrak{g})_{\eta_0} - \text{mod}$.

2. Lecture 2: Formal deformation theory and Kontsevich's theorem

In this section, we will often work with dg, i.e., differential graded algebras. These algebras have *homological* grading, and they are superalgebras with the Koszul sign rule, i.e., the permutation of tensors $v \otimes w \mapsto w \otimes v$ carries the additional sign $(-1)^{|v||w|}$. In particular, a dg commutative algebra has the commutativity rule

$$v \cdot w = (-1)^{|v||w|} w \cdot v,$$

and a dg Lie algebra has the anticommutativity rule

$$\{v, w\} = -(-1)^{|v||w|}\{w, v\}.$$

This is not so important in this lecture, but will become more important in Lecture 3.

2.1. Definition of Hochschild (co)homology.

Let A be an associative algebra. The Hochschild (co)homology is the natural (co)homology theory associated to associative algebras. We give a convenient definition in terms of Ext and Tor. Let $A^e := A \otimes_{\mathbf{k}} A^{\mathrm{op}}$, where A^{op} is the opposite algebra, defined to be the same underlying vector space as A, but with the opposite multiplication, $a \cdot b := ba$. Note that A^e-modules are the same as A-bimodules (where, by definition, $\mathbf{k}$ acts the same on the right and the left, i.e., by the fixed $\mathbf{k}$-vector space structure on the bimodule).

Definition 2.1. Define the Hochschild homology and cohomology, respectively, of A, with coefficients in an A-bimodule M, by

$$\mathsf{HH}_i(A, M) := \operatorname{Tor}_i^{A^e}(A, M), \quad \mathsf{HH}^i(A, M) := \operatorname{Ext}^i_{A^e}(A, M).$$

Without specifying the bimodule M, we are referring to $M = A$.

Moreover, note that, by definition, the Hochschild cohomology $\mathsf{HH}^\bullet(A)$ is a *ring* under the Yoneda product of extensions (in fact, it is graded commutative). It also has a richer structure, that of a Gerstenhaber algebra (essentially an odd Poisson bracket), that we will introduce and use later on.

The most important object above for us will be $\mathsf{HH}^2(A)$, in accordance with the (imprecise) principle:

(2.2) The space $\mathsf{HH}^2(A)$ parameterizes all (infinitesimal, filtered, or formal) deformations of A, up to obstructions (in $\mathsf{HH}^1(A)$) and equivalences (in $\mathsf{HH}^3(A)$).

More generally, given any type of algebra structure on an ordinary (not dg) vector space B:

The space $H^2_{\mathcal{O}}(B,B)$ parameterizes (infinitesimal, filtered, or formal) deformations of the $\mathcal{O}$-algebra structure on B, up to obstructions (in $H^3_{\mathcal{O}}(B,B)$) and equivalences (in $H^1_{\mathcal{O}}(B,B)$).

We will apply this to Poisson algebras: $\mathsf{HP}^2(B,B)$ parameterizes Poisson deformations of a Poisson algebra B, up to the aforementioned caveats.

Remark 2.3. Stated this way, while convenient, one misses that Hochschild (co)homology is the natural theory attached to associative algebras, in the general sense that one can attach a natural homology theory to other types of algebras, e.g., algebras over an arbitrary operad (such as the Lie or commutative operads, rather than the associative one), where the correct definition not obviously a generalization of the above. For instance, for general types of algebras, their category of modules does not form an abelian category, but only a model category.

In the case of Lie algebras $\mathfrak{g}$, the category of $\mathfrak{g}$-modules is in fact abelian, and equivalent to the category of $U\mathfrak{g}$-modules. Let $H^\bullet_{\mathrm{Lie}}(\mathfrak{g},-)$ denotes the Lie (or Chevalley-Eilenberg) cohomology of $\mathfrak{g}$ with coefficients in Lie modules. One then has (cf. [Wei94, Exercise 7.3.5]) $\mathrm{Ext}^\bullet_{U\mathfrak{g}}(M,N) \cong H^\bullet_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(M,N))$. So $H^\bullet_{\mathrm{Lie}}(\mathfrak{g},N) \cong \mathrm{Ext}^\bullet_{U\mathfrak{g}}(\mathbf{k},N)$, the extensions of $\mathbf{k}$ by N as $\mathfrak{g}$-modules; this differs from the associative case where one considers the extensions of A, rather than $\mathbf{k}$, by N. (Note that, since A need not be augmented, one does not necessarily have a module $\mathbf{k}$ of A anyway. Also note that, for associative algebras, bimodules are the correct notion of modules over an algebra that are analogous to Lie modules over Lie algebras.)

2.2. **Formality theorems.** We will also state results for C^∞ manifolds, both for added generality, and also to help build intuition. In the C^∞ case we will take $\mathbf{k} = \mathbf{C}$ and let $\mathcal{O}_X$ be the algebra of smooth complex-valued functions on X. There is one technicality: we restrict to *local* Hochschild cohomology, cf. Remark 3.7. In the affine case, in this section, we will assume that $\mathbf{k}$ is a field of characteristic zero.

Theorem 2.4 (Hochschild-Kostant-Rosenberg)**.** *Let X be a either smooth affine variety over a field $\mathbf{k}$ of characteristic zero or a C^∞ manifold. Then, the Hochschild cohomology ring $\mathsf{HH}^\bullet(\mathcal{O}_X,\mathcal{O}_X) \cong \wedge^\bullet_{\mathcal{O}_X} T_X$ is the ring of* polyvector fields *on X.*

Explicitly, a polyvector field of degree d is a sum of elements of the form

$$\xi_1 \wedge \cdots \wedge \xi_d,$$

where each $\xi_i \in T_X$ is a vector field on X, i.e., when X is an affine algebraic variety, $T_X = \mathrm{Der}(\mathcal{O}_X,\mathcal{O}_X)$.

Thus, the deformations are classified by certain *bivector fields* (those whose obstructions vanish). As we will explain, the first obstruction is that the bivector field be Poisson. Moreover, deforming along this direction is the same as quantizing the Poisson structure. So the question reduces to: which Poisson structures on $\mathcal{O}_X$ can be quantized? It turns out, by a very deep result of Kontsevich, that *they all can.*

To make this precise, we need to generalize the notion of quantization: we spoke about quantizing graded $\mathcal{O}_X$, but in general this is not graded when X is smooth. (Indeed, a grading would mean that X has a $\mathbf{G}_m$-action, and in general the fixed point(s) will be singular. For example, if $\mathcal{O}_X$ is nonnegatively graded with $(\mathcal{O}_X)_0 = \mathbf{k}$, i.e., the action is contracting with a single fixed point, then X is singular unless $X = \mathbf{A}^n$ is an affine space.)

Instead, we introduce a formal parameter $\hbar$:

Definition 2.5. A *deformation quantization* is a one-parameter formal deformation $A_\hbar = (B[\![\hbar]\!], \star)$ of a Poisson algebra B which is compatible with the Poisson structure:

(2.6) $$a \star b \equiv ab \pmod{\hbar}, \forall a, b \in B, a \star b - b \star a \equiv \hbar\{a, b\} \pmod{\hbar^2}, \forall a, b \in B.$$

We will often use the observation that, for X an affine variety or a smooth manifold, a Poisson structure on $\mathcal{O}_X$ is the same as a bivector field $\pi \in \wedge^2 T_X$,

$$\{f, g\} = \pi(df \wedge dg),$$

satisfying the Jacobi identity. In terms of π, the Jacobi identity says $[\pi, \pi] = 0$, using the Schouten-Nijenhuis bracket.

Example 2.7. The simplest example is the case $X = \mathbf{A}^n$ with a constant Poisson bivector field, which can always be written up to choice of basis as

$$\pi = \sum_{i=1}^{m} \partial_{x_i} \wedge \partial_{y_i}, \quad \text{i.e.,} \quad \{f, g\} = \sum_{i=1}^{m} \partial_{x_i} f \partial_{y_i} g - \partial_{y_i} f \partial_{x_i} g,$$

where $2m \leq n$. Then there is a well-known deformation quantization, called the Moyal-Weyl star product:

$$f \star g = \mu \circ e^{\frac{1}{2}\hbar\pi}(f \otimes g), \quad \mu(a \otimes b) := ab.$$

When $2m = n$, so that the Poisson structure is symplectic, this is actually isomorphic to the usual Weyl quantization: see the next exercise.

Exercise 2.8. (a) Show that, for the Moyal-Weyl star product,

$$x_i \star y_j - y_j \star x_i = \hbar\delta_{ij}, \quad x_i \star x_j = x_j \star x_i, \quad y_i \star y_j = y_j \star y_i.$$

Conclude that the map which is the identity on linear functions V on $\mathbf{A}^n = \mathbf{A}^{2m} \times \mathbf{A}^{n-2m}$ defines an isomorphism

$$A_m \otimes \mathcal{O}_{\mathbf{A}^{n-2m}} \xrightarrow{\sim} (cO_{\mathbf{A}^n}[\![\hbar]\!], \star),$$

where A_m is the Weyl algebra on $2m$ variables.

(b) Show that the symmetrization map,

$$v_1 \cdots v_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} v_{\sigma(1)} \star \cdots \star v_{\sigma(k)},$$

for $v_1, \ldots, v_k \in V$ linear functions on $\mathbf{A}^n$, inverts the isomorphism of (a).

(c) Show that the Moyal-Weyl star product is the unique star product which quantizes the constant Poisson structure $\pi = \sum_{i=1}^m \partial_{x_i} \wedge \partial_{y_i}$ such that $a \star b$ is a linear combination of contractions of tensor powers of π with a and b, which is linear in a and b (but *not* necessarily in π!). Take a look at §2.3 below for a more explicit description of all such operators, as a linear combination of operators associated to certain graphs.

(d) If you know Weyl's fundamental theorem of invariant theory, conclude from (c) that the functor from constant Poisson brackets to their Moyal-Weyl star products is the unique $\mathsf{GL}(V)$-equivariant functor assigning each constant Poisson bracket a deformation quantization.

We will also need the notion of a *formal Poisson deformation* of a Poisson algebra $\mathcal{O}_X$: this is a $\mathbf{k}[\![\hbar]\!]$-linear Poisson bracket on $\mathcal{O}_X[\![\hbar]\!]$ which reduces modulo $\hbar$ to the original Poisson bracket.

Theorem 2.9. [Kon03, Kon01, Yek05, DTT07] Every Poisson structure on a smooth affine variety over a field of characteristic zero, or on a C^∞-manifold, admits a canonical deformation quantization. In particular, every graded Poisson algebra with Poisson bracket of degree $-d < 0$ admits a filtered quantization.

Moreover, there is a canonical bijection, up to isomorphisms equal to the identity modulo $\hbar$, between deformation quantizations and formal Poisson deformations.

Remark 2.10. As shown by O. Mathieu [Mat97], in general there are obstructions to the existence of a quantization. We explain this following §1.4 of `www.math.jussieu.fr/~keller/emalca.pdf`, which more generally forms a really nice reference for much of the material discussed in these notes!

Namely, let $\mathfrak{g}$ is a Lie algebra over a field $\mathbf{k}$ of characteristic zero such that $\mathfrak{g} \otimes_{\mathbf{k}} \bar{\mathbf{k}}$ is simple and not isomorphic to $\mathfrak{sl}_n(\bar{\mathbf{k}})$ for any n, where $\bar{\mathbf{k}}$ is the algebraic closure of $\mathbf{k}$. (In the cited references, one takes $\mathbf{k} = \mathbf{R}$ and thus $\bar{\mathbf{k}} = \mathbf{C}$, but this assumption is not needed.)

One then considers the Poisson algebra $B := \mathsf{Sym}\, \mathfrak{g}/(\mathfrak{g}^2)$, equipped with the Lie bracket on $\mathfrak{g}$. In other words, this is the quotient of $\mathcal{O}_{\mathfrak{g}^*}$ by the square of the augmentation ideal, i.e., the maximal ideal of the origin. In particular, $\mathsf{Spec}\, B$ is the first infinitesimal neighborhood of the origin in $\mathfrak{g}^*$, so it is nonreduced, and set-theoretically just a point, albeit with a nontrivial Poisson structure.

We claim that B does not admit a deformation quantization. For a contradiction, suppose it did admit one, $B_\hbar := (B[\![\hbar]\!], \star)$. Since $B = \mathbf{k} \oplus \mathfrak{g}$ has the property that $B \otimes_{\mathbf{k}} \bar{\mathbf{k}}$ is a semisimple Lie algebra, it follows that $H^2_{\mathrm{Lie}}(B) = 0$, i.e., B has no deformations as a Lie algebra. Hence, $B_\hbar \cong B[\![\hbar]\!]$ as a Lie algebra. Now let $K := \overline{\mathbf{k}(\!(\hbar)\!)}$, an algebraically closed field, and set $\tilde{B} := B_\hbar \otimes_{\mathbf{k}[\![\hbar]\!]} K$. Since this is finite-dimensional over the algebraically closed field K, Wedderburn theory implies that, as a K-algebra, $\tilde{B} \cong M \oplus J$, where M is a product of matrix rings of various sizes and J is nilpotent. Thus, as a Lie algebra over K, the only simple quotients of $\tilde{B}$ are of the form $\mathfrak{sl}_n(K)$. On the other hand, $\tilde{B} = (\mathbf{k} \oplus \mathfrak{g}) \otimes_{\mathbf{k}} K$ as a Lie algebra, which has the simple quotient $\mathfrak{g} \otimes_{\mathbf{k}} K$, which is not of the form $\mathfrak{sl}_n(K)$ by assumption. This is a contradiction.

Example 2.11. In the case $X = \mathbf{A}^n$ with a constant Poisson bivector field, Kontsevich's star product coincides with the Moyal-Weyl one. (This actually follows from Exercise 2.8.(d); one can also verify explicitly that the two formulas coincide—perhaps the only case where Kontsevich's formula is tractable! This is a good exercise!)

Example 2.12. Next, let $\{-,-\}$ be a linear Poisson bracket on $\mathfrak{g}^* = \mathbf{A}^n$, i.e., a Lie bracket on the vector space $\mathfrak{g} = \mathbf{k}^n$ of linear functions. As explained in [Kon03] (see Exercise 1 of Exercise Sheet 4), if $(\mathcal{O}_{\mathbf{A}^n}[\![\hbar]\!], \star)$ is Kontsevich's canonical quantization, then

$$x \star y - y \star x = [x, y],$$

so that, as in Exercise 2.8, the map which is the identity on linear functions yields an isomorphism

$$U_\hbar(\mathfrak{g}) \to (\mathcal{O}_{\mathfrak{g}^*}[\![\hbar]\!], \star).$$

It turns out, by [Dit99], that, up to applying a gauge equivalence to Kontsevich's star-product (see §4.4 below), the inverse to this is again the symmetrization map of Exercise 2.8.(b). The latter star-product is called the Gutt product, and is very complicated to write out explicitly (it probably requires the Campbell-Baker-Hausdorff formula). For a description of this product, see, e.g., [Dit99, (13)]. Moreover, as first noticed in [Arn98] (see also [Dit99]), there is no gauge equivalence required when $\mathfrak{g}$ is nilpotent, i.e., in this case the Kontsevich star-product equals the Gutt product (this is essentially because nothing else can happen in this case, similarly to Exercise 2.8.(d)).

These theorems all rest on the basic statement that the Hochschild cohomology of a smooth affine variety or C^∞-manifold is *formal*, i.e., the Hochschild cocomplex (which computes its cohomology), is equivalent to its cohomology not merely as a vector space, but as dg Lie algebras up to homotopy.

(In fact, this statement can be made to be equivalent to the bijection of Theorem 2.9 if one extends to deformations over dg commutative rings rather than merely formal power series.)

I will explain what formality of dg Lie algebras means precisely later. For now, let me give some simpler examples of formality. First of all, note that all *complexes of vector spaces are always equivalent to their cohomology*, i.e., they are all formal. This follows because, given a complex $C^\bullet$, one can always find an *isomorphism of complexes*

$$C^\bullet \xrightarrow{\sim} H^\bullet(C^\bullet) \oplus S^\bullet,$$

where $H^\bullet$ is the homology of $C^\bullet$, and $S^\bullet$ is a contractible complex.

But if we consider modules over a more general ring that is not a field, this is no longer true. Consider, for example, the complex

$$0 \to \mathbf{Z} \xrightarrow{\cdot 2} \mathbf{Z} \to 0.$$

The homology is $\mathbf{Z}/2$, but it is impossible to write the complex as a direct sum of $\mathbf{Z}/2$ with a contractible complex, since $\mathbf{Z}$ has no torsion. That is, the above complex is *not* formal.

Now, the subtlety with the formality of Hochschild cohomology is that, even though the underlying Hochschild cochain complex is automatically formal as a *complex of vector spaces*, it is *not* automatically formal as *dg Lie algebras*. For example, it may not necessarily be isomorphic to a direct sum of its cohomology and another dg Lie algebra (although being formal does not require this, but only that the dg Lie algebra be "homotopy equivalent" to its cohomology).

The statement that the Hochschild cochain complex is formal is *stronger* than merely the existence of deformation quantizations. It implies, for example:

Theorem 2.13. If (X, ω) is either an affine symplectic variety over a field of characteristic zero or a symplectic C^∞ manifold, and $A = (\mathcal{O}_X[\![\hbar]\!], \star)$ is a deformation quantization of X, then $\mathsf{HH}^\bullet(A[\hbar^{-1}]) \cong H^\bullet_{DR}(X, \mathbf{k}(\!(\hbar)\!))$, and there is a universal formal deformation of $A[\hbar^{-1}]$ over the base $H^2_{DR}(X)$.

Namely, if $\omega_1, \ldots, \omega_m \in \Omega^2_X$ is a basis of $H^2_{DR}(X)$, then the universal family consists of the algebras $A'[\hbar^{-1}]$ where A' is a deformation quantization corresponding via Theorem 2.9 to a formal Poisson bracket in $(\omega + \hbar \cdot \langle \omega_1, \ldots, \omega_m \rangle[\![\hbar]\!])^{-1}$.

2.3. **Description of Kontsevich's deformation quantization for $\mathbf{R}^d$.** It is worth explaining the general form of the star-products given by Kontsevich's theorem for $X = \mathbf{R}^d$, considered either as a smooth manifold or an affine algebraic variety over $\mathbf{k} = \mathbf{R}$. This is taken from [Kon03, §2]; the interested reader is recommended to look there for details (and an example with a drawn graph).

Suppose we are given a Poisson bivector $\pi \in \wedge^2 T_X$. Then Kontsevich's star product $f \star g$ is a linear combination of all possible ways of applying π multiple times to f and g (cf. Exercise 2.8), with very sophisticated weights.

The possible ways of applying π are easy to describe using directed graphs. Namely, the graphs we need to consider are placed in the closed upper-half plane $\{(x, y) \mid y \geq 0\} \subseteq \mathbf{R}^2$, satisfying the following properties:

(1) There are exactly two vertices along the x-axis, labeled by L and R. The other vertices are labeled $1, 2, \ldots, m$.
(2) L and R are sinks, and all other vertices have exactly two outgoing edges.
(3) At every vertex j, the two outgoing edges should be labeled by the symbols e^j_1 and e^j_2. That is, we fix an ordering of the two edges and denote them by e^j_1 and e^j_2.
(4) These two vertices are sinks. The other vertices should have exactly two outgoing edges.

Write our Poisson bivector π in coordinates as

$$\pi = \sum_{i<j} \pi^{i,j} \partial_i \wedge \partial_j.$$

Let $\pi^{j,i} := -\pi^{i,j}$. We attach to the graph Γ a bilinear differential operator $B_{\Gamma,\pi} : \mathcal{O}_X^{\otimes 2} \to \mathcal{O}_X$, as follows. Let $E_\Gamma = \{e_1^1, e_1^2, \ldots, e_m^1, e_m^2\}$ be the set of edges. Given an edge $e \in E_\Gamma$, let $t(e)$ denote the target vertex of e. Then

$$B_{\Gamma,\pi}(f \otimes g) := \sum_{I:E_\Gamma \to \{1,\ldots,d\}} \Bigl[\prod_{i=1}^{m} \Bigl(\prod_{e \in E_\Gamma | t(e)=i} \partial_{I(e)} \Bigr) \pi^{I(e_i^1),I(e_i^2)} \Bigr] \cdot \Bigl(\prod_{e \in E_\Gamma | t(e)=L} \partial_{I(e)} \Bigr)(f) \cdot \Bigl(\prod_{e \in E_\Gamma | t(e)=R} \partial_{I(e)} \Bigr)(g).$$

Now, let V_Γ denote the set of vertices of Γ, so that in the above formula, $m = |\Gamma|$. Then the star product is given by

$$f \star g = \sum_\Gamma (\hbar/2)^{|\Gamma|} W_\Gamma B_{\Gamma,\pi}(f \otimes g),$$

where we sum over isomorphism classes of graphs satisfying conditions (1)–(4) above (we only need to include one for each isomorphism class forgetting the labeling, since the operator is the same up to a sign). The $W_\Gamma \in \mathbf{R}$ are weights given by very explicit integrals (which in general are impossible to evaluate). Note that, to be a quantization, for Γ_0 the graph with no edges (and thus only vertices L and R), $W_\Gamma = 1$.

Important idea / observation: Linear combinations of operators $B_{\Gamma,\pi}$ as above are exactly all of bilinear operators obtainable by contracting tensor powers of π with f and g, cf. Exercise 2.8.(c).

2.4. **Formal deformations of algebras.** Now we define more precisely formal deformations. These generalize star products to the case of deforming arbitary associative algebras, and also to the setting where more than one deformation parameter is allowed. Generally, we are interested in commutative augmented rings $R \supseteq R_+$ such that R is complete with respect to the R_+-adic topology, i.e., such that $R = \lim_{\leftarrow} R/R_+^m$. We call such rings *complete augmented* commutative rings. We will need the completed tensor product,

$$A \hat{\otimes} R = \lim_{\leftarrow} A \otimes R/R_+^m.$$

Example 2.14. When R is a formal power series ring $R = \mathbf{k}[\![t_1, \ldots, t_n]\!]$,

$$A \hat{\otimes} R = A[\![t_1, \ldots, t_n]\!] = \{ \sum_{i_1,\ldots,i_n \geq 0} a_{i_1,\ldots,i_n} t_1^{i_1} \cdots t_n^{i_n} \mid a_{i_1,\ldots,i_n} \in A\}.$$

Definition 2.15. A formal deformation of A over a commutative complete augmented ring R is an R-algebra A' isomorphic to $A \hat{\otimes}_{\mathbf{k}} R$ as an R-module such that $A' \otimes_R R_+ \cong A$ as a $\mathbf{k}$-algebra.

Equivalently, a formal deformation is an algebra $(A \hat{\otimes}_{\mathbf{k}} R, \star)$ such that $a \star b \equiv ab \pmod{R_+}$.

Example 2.16. If $R = \mathbf{k}[\![t_1, \ldots, t_n]\!]$, then a formal deformation of A over R is the same as an algebra $(A[\![t_1, \ldots, t_n]\!], \star)$ such that $a \star b \equiv ab \pmod{t_1, \ldots, t_n}$.

We will often restrict our attention to one-parameter deformations, and in this case, the parameter is often denoted by $\hbar$ (which originates from quantum physics).

2.5. **Formal vs. filtered deformations.** Main idea: if A is a graded algebra, we can consider filtered deformations on the same underlying filtered vector space A. These are equivalent to homogeneous formal deformations of A, by replacing relations of degree d, $\sum_{i=0}^{d} p_i = 0$ with $|p_i| = i$, by $\sum_i \hbar^{d-i} p_i = 0$, which are now homogeneous with the sum of the grading on A and $|\hbar| = 1$.

We begin with two motivating examples:

Example 2.17. We can form a homogenized version of the Weyl algebra:

$$\mathsf{Weyl}_\hbar(V) = TV[\![\hbar]\!]/(vw - wv - \hbar(v,w)).$$

Then this is homogeneous with $|\hbar| = 2$.

Example 2.18. The homogenized universal enveloping algebra $U_\hbar\mathfrak{g}$ is given by

$$U_\hbar\mathfrak{g} = T\mathfrak{g}[\![\hbar]\!]/(xy - yx - \hbar[x,y]),$$

with $|\hbar| = 1$.

We now proceed to precise definitions and statements:

Definition 2.19. Let A be an increasingly filtered associative algebra. Then, the *Rees algebra* of A is the graded algebra

$$A^r := \bigoplus_{i\in\mathbf{Z}} A_{\le i}\cdot\hbar^i,$$

equipped with the multiplication

$$(a\cdot\hbar^i)\cdot(b\cdot\hbar^j) = ab\cdot\hbar^{i+j}.$$

Similarly, let the *completed* Rees algebra be

$$\hat{A} := \{\sum_{i=m}^{\infty} a_i\cdot\hbar^i \mid a_i\in A_{\le i}, m\in\mathbf{Z}\}.$$

The Rees algebra A^r is now a *graded* algebra, by $|\hbar| = 1$ and $|A| = 0$; its completion with respect to the ideal $(\hbar)$ is $\hat{A}$. This actually defines an equivalence, which we state as follows:

Lemma 2.20. The functor $A \mapsto A^r$ defines an equivalence of categories from increasingly filtered $\mathbf{k}$-algebras to nonnegatively graded free $\mathbf{k}[\hbar]$-algebras, with quasi-inverse $C \mapsto C/(\hbar-1)$ obtained by setting $\hbar = 1$, and assigning the filtration $(C/(\hbar-1))_{\le m}$ to be the image of those elements of degree $\le m$ in C.

The proof is left as an exercise. In other words, the equivalence replaces $A_{\le m}$ with the homogeneous part $(A^r)_m$.

Remark 2.21. The Rees algebra can be viewed as interpolating between A and $\mathsf{gr}\,A$, in that $A^r[\hbar^{-1}] = A[\hbar,\hbar^{-1}]$ and $A^r/(\hbar) = \mathsf{gr}\,A$. Similarly, $\hat{A}[\hbar^{-1}] = A(\!(\hbar)\!)$ and $\hat{A}/(\hbar) = \mathsf{gr}\,A$. So over any point other than the origin of $\mathsf{Spec}\,\mathbf{k}[\hbar]$ (or over the generic point of $\mathsf{Spec}\,\mathbf{k}[\![\hbar]\!]$), we recover A (or $A(\!(\hbar)\!)$), and at the origin (or the special point of $\mathsf{Spec}\,\mathbf{k}[\![\hbar]\!]$), we obtain $\mathsf{gr}\,A$.

Given a graded algebra A, we canonically obtain a filtered algebra by $A_{\le n} = \bigoplus_{m\le n} A_m$. In this case, we can consider filtered deformations of A of the form $(A,\star_f)$, where $\star_f$ reduces to the usual product on $\mathsf{gr}\,A = A$, i.e., $\mathsf{gr}(A,\star_f) = A$.

Corollary 2.22.
(i) Let A be a graded algebra. Then there is an equivalence between filtered deformations $(A,\star_f)$ and formal deformations $(A[\![\hbar]\!],\star)$ which are graded with respect to the sum of the grading on A and $|\hbar| = 1$, given by $(A,\star_f) \mapsto \widehat{(A,\star_f)}$. For the opposite direction, because of the grading, $(A[\![\hbar]\!],\star) = (A[\hbar],\star)\otimes_{\mathbf{k}[\hbar]}\mathbf{k}[\![\hbar]\!]$, so one can take $(A[\hbar],\star)/(\hbar-1)$.
(ii) Let B be a graded Poisson algebra with Poisson bracket of degree $-d$. Then there is an equivalence between filtered quantizations $(B,\star_f)$ and deformation quantizations $(B[\![\hbar]\!],\star)$ which are homogeneous with respect to the sum of the grading on B and $|\hbar| = d$, also given by $(B,\star_f) \mapsto \widehat{(B,\star_f)}$. For the opposite direction, as in (i), one can take $(B[\hbar],\star)/(\hbar-1)$.

2.6. **Universal deformations.**

Definition 2.23. A versal formal deformation $(A\hat{\otimes}R, \star_u)$, for $R = \mathbf{k}[\![t_1, \ldots, t_n]\!]$, is a formal deformation such that, for every one-parameter formal deformation $(A[\![\hbar]\!], \star)$, there exists a continuous homomorphism $p : R \to \mathbf{k}[\![\hbar]\!]$ such that $a \star b = p(a \star_u b)$.

Similarly, a versal filtered deformation of a graded algebra A is a graded associative algebra $(A[t_1, \ldots, t_n], \star_u)$ with the sum of the grading on A with gradings $|t_1|, \ldots, |t_n| \geq 1$ such that, for every filtered deformation $(A, \star)$ of A, there is a homomorphism $p : \mathbf{k}[t_1, \ldots, t_n] \to \mathbf{k}$ such that $a \star b = p(a \star_u b)$.

The deformation is *universal* if the homomorphism p is unique.

Remark 2.24. The relationship between $(A[\![\hbar]\!], \star)$ and the (uni)versal $(A\hat{\otimes}R, \star_u)$ can be stated more geometrically as follows: p is a formal $\mathsf{Spf}\,\mathbf{k}[\![\hbar]\!]$-point p of $\mathsf{Spf}\,R$, and $(A[\![\hbar]\!], \star)$ is the pullback of $(A\hat{\otimes}R, \star_u)$, namely, $(A[\![\hbar]\!], \star) = (A\hat{\otimes}R, \star_u) \otimes_R p$.

The principle that $\mathsf{HH}^3(A)$ classifies obstructions (which will be made precise in §3.7 below) leads to the following important result, which is provable using the Maurer-Cartan formalism discussed below:

Proposition 2.25. If $\mathsf{HH}^3(A) = 0$, then there exists a versal formal deformation of A with base $\mathbf{k}[\![\mathsf{HH}^2(A)]\!]$. If, furthermore, $\mathsf{HH}^1(A) = 0$, then this is a universal deformation.

In the case when A is filtered, then if $\mathsf{HH}^3(A)_{\leq 0} = 0$, there exists a versal filtered deformation of A with base $\mathbf{k}[HH^2(A)_{\leq 0}]$. If, furthermore, $\mathsf{HH}^1(A)_{\leq 0} = 0$, then this is a universal filtered deformation.

Exercise 2.26. Prove the proposition!

Hint: Use the Maurer-Cartan formalism, and the fact that, if $C^\bullet$ is an arbitrary complex of vector spaces, there exists a homotopy $h : C^\bullet \to C^{\bullet-1}$ such that $C^\bullet \cong H^\bullet(C) \oplus (hd + dh)(C^\bullet(C))$, i.e., $\mathrm{Id} - (hd + dh)$ is a projection of $C^\bullet$ onto a subspace isomorphic to its homology. In this case, let $i : H^\bullet(C) \hookrightarrow C^\bullet$ be the obtained inclusion.

In the case $\mathsf{HH}^3(A) = 0$, let h be a homotopy as above for $C(A)$ and $i : \mathsf{HH}^\bullet(A) \to C^\bullet(A)$ the obtained inclusion. Show that a formula for a versal deformation $(A[\![\mathsf{HH}^2(A)]\!], \star)$, in the case $\mathsf{HH}^3(A) = 0$, can be given by the formal function $\Gamma \in C(A)[\![\mathsf{HH}^2(A)]\!]$,

$$\Gamma := (\mathrm{Id} + h \cdot MC)^{-1} \circ i,$$

where $MC(x) := dx + \frac{1}{2}[x, x]$ is the LHS of the Maurer-Cartan equation (4.13). More precisely, plugging in any power series $\sum_{m \geq 1} \hbar^m \cdot \gamma_m \in \hbar \cdot \mathsf{HH}^2(A)[\![\hbar]\!]$, one obtains the Maurer-Cartan element $\sum_{m \geq 1} \hbar^m \Gamma(\gamma_m) \in \hbar \cdot C^2(A)[\![\hbar]\!]$, and the associated star products yield all possible formal deformations of A up to gauge equivalence.

Example 2.27. If $A = \mathsf{Weyl}(V) \rtimes G$ for $G < \mathsf{Sp}(V)$ a finite subgroup, then A has finite Hochschild dimension, and by [AFLS00], $\mathsf{HH}^i(A)$ is the space of conjugation-invariant functions on the set of group elements $g \in G$ such that $\mathrm{rk}(g - \mathrm{Id}) = i$. In particular, $\mathsf{HH}^i(A) = 0$ when i is odd, and $\mathsf{HH}^2(A)$ is the space of conjugation-invariant functions on the set of *symplectic reflections*: those elements fixing a codimension-two symplectic hyperplane. Moreover, $\mathsf{HH}^2(A) = \mathsf{HH}^2(A)_{\leq -2}$ is all in degree -2 (the degree also of the Poisson bracket on $\mathcal{O}_V$). Thus, there is a universal filtered deformation of A parameterized by elements $c \in \mathsf{HH}^2(A)$ as above. Let S be the set of symplectic reflections, i.e., elements such taht $\mathrm{rk}(g - \mathrm{Id}) = 2$. Then $c \in \mathbf{k}[S]^G$. This deformation is the *symplectic reflection algebra* $H_{1,c}(G)$ [EG02], first constructed by Drinfeld. Explicitly, this deformation is presented by

$$TV/\Big(xy - yx - \omega(x, y) + 2\sum_{s \in S} c(s)\omega_s(x, y) \cdot s\Big),$$

where ω_s is the composition of ω with the projection to the sum of the nontrivial eigenspaces of s (which is a two-dimensional symplectic vector space). In other words, ω_s is the restriction of the symplectic form ω to the two-dimensional subspace orthogonal to the symplectic reflecting hyperplane of s.

Example 2.28. If X is a smooth affine variety or smooth C^∞ manifold, and G is a group acting by automorphisms on X, then by [Eti04], $\mathsf{HH}^2(\mathcal{D}_X \rtimes G) \cong H^2_{DR}(X)^G \oplus \mathbf{k}[S]^G$, where S is the set of pairs (g, Y) where $g \in G$ and $Y \subseteq X^g$ is a connected (hence irreducible) subvariety of codimension one. Then $\mathbf{k}[S]^G$ is the space of $\mathbf{k}$-valued functions on S which are invariant under the action of G, $h \cdot (g, Y) = (hgh^{-1}, h(Y))$. Furthermore, by [Eti04], all deformations are *unobstructed* and there exists a universal filtered deformation $H_{1,c,\omega}(X)$ parameterized by $c \in \mathbf{k}[S]^G$ and $\omega \in H^2_{DR}(X)^G$; the Rees algebra construction (or restriction to a formal neighborhood of the origin of the parameter space $(H^2_{DR}(X)^G \oplus \mathbf{k}[S]^G)$ yields a universal formal deformation.

3. Lecture 3: Hochschild cohomology and infinitesimal deformations

3.1. **The bar resolution.** To compute Hochschild (co)homology using Definition 2.1, one can resolve A as an A^e-module, i.e., A-bimodule. The standard way to do this is via the *bar resolution*:

Definition 3.1. The bar resolution, $C^{\text{bar}}(A)$, of an associative algebra A over $\mathbf{k}$ is the complex

$$\cdots \longrightarrow A \otimes A \otimes A \longrightarrow A \otimes A \twoheadrightarrow A, \tag{3.2}$$

$$(a \otimes b \otimes c) \longmapsto ab \otimes c - a \otimes bc,$$

$$a \otimes b \longmapsto ab.$$

More conceptually, we can define the complex as

$$T_A(A \cdot \epsilon \cdot A), \quad d = \partial_\epsilon,$$

viewing $A \cdot \epsilon \cdot A$ as an A-bimodule, assigning degrees by $|\epsilon| = 1, |A| = 0$, and viewing the differential $d = \partial_\epsilon$ is a *graded derivation* of degree -1, i.e.,

$$\partial_\epsilon(x \cdot y) = \partial_\epsilon(x) \cdot y + (-1)^{|y|} x \cdot \partial_\epsilon(y).$$

Finally, $\partial_\epsilon(\epsilon) = 1$ and $\partial_\epsilon(A) = 0$.

3.2. **The Hochschild (co)homology complexes.** Using the bar resolution, we conclude that

$$\mathsf{HH}_i(A, M) = \mathrm{Tor}_i^{A^e}(A, M) = H_i(C^{\text{bar}}_\bullet(A) \otimes_{A^e} M) = H_i(C_\bullet(A, A)), \tag{3.3}$$

where

$$C_\bullet(A, M) := M \otimes_A C^{\text{bar}}_\bullet(A) \otimes_A M / [A, M \otimes_A C^{\text{bar}}_\bullet(A)] \tag{3.4}$$

$$= \cdots \longrightarrow M \otimes A \otimes A \longrightarrow M \otimes A \longrightarrow A,$$

$$(m \otimes b \otimes c) \longmapsto mb \otimes c - m \otimes bc + cm \otimes b,$$

$$a \otimes b \longmapsto ab - ba.$$

Similarly,

$$\mathsf{HH}^i(A, M) = \mathrm{Ext}^i_{A^e}(A, M) = H^i(C^\bullet(A, M)), \tag{3.5}$$

where

$$\begin{aligned} C^\bullet(A, M) &= \mathrm{Hom}_{A^e}(C^{\mathrm{bar}}_\bullet(A), M)) \\ &= \cdots \longleftarrow \mathrm{Hom}_{\mathbf{k}}(A \otimes A, M) \longleftarrow \mathrm{Hom}_{\mathbf{k}}(A, M) \longleftarrow M, \\ &\qquad \phi(a \otimes b) = a\phi(b) - \phi(ab) + \phi(a)b \longleftarrow\!\dashv \phi, \\ &\qquad\qquad \phi(a) = ax - xa \longleftarrow\!\dashv x. \end{aligned} \tag{3.6}$$

As for cohomology, we will denote $C^\bullet(A) := C^\bullet(A, A)$.

Remark 3.7. To extend Hochschild cohomology $HH^\bullet(A)$ to the case $A = \mathcal{O}_X := C^\infty(X)$ for X a C^∞-manifold, we define $C^\bullet(A)$ as above, except restricting each space $\mathrm{Hom}_{\mathbf{k}}(A^{\otimes m}, A)$ to the subspace of *smooth polydifferential operators*, i.e., linear maps spanned by tensor products of smooth (rather than algebraic) differential operators $A \to A$. These are operators which are spanned by polynomials in smooth vector fields (which are not the same as algebraic derivations in the smooth setting: a smooth vector field is always a derivation of A as an abstract algebra, but not conversely). We will not define more general Hochschild (co)homology spaces in the C^∞ case.

3.3. **Zeroth Hochschild homology.** We see from the above that $\mathsf{HH}_0(A) = A/[A, A]$, where the quotient is taken as a vector space.

Example 3.8. Suppose that $A = TV$ is a free algebra on a vector space V. Then $\mathsf{HH}_0(A) = TV/[TV, TV]$ is the vector space of *cyclic words* in V.

Similarly, if $A = \mathbf{k}Q$ is the path algebra of a quiver Q, then $\mathsf{HH}_0(A) = \mathbf{k}Q/[\mathbf{k}Q, \mathbf{k}Q]$ is the vector space of *cyclic paths* in the quiver Q (which by definition do not have an initial or terminal vertex).

Example 3.9. If A is commutative, then $\mathsf{HH}_0(A) = A$, since $[A, A] = 0$.

3.4. **Zeroth Hochschild cohomology.** For the rest of the section, we will only need Hochschild cohomology, not homology, so when we write $C(A)$ (omitting the $\bullet$), we mean the Hochschild *cohomology* ring.

Note that $\mathsf{HH}^0(A) = \{a \in A \mid ab - ba = 0, \forall b \in A\} = Z(A)$, the center of the algebra A.

Example 3.10. If $A = TV$ is a tensor algebra, then $\mathsf{HH}^0(A) = \mathbf{k}$: only scalars are central. The same is true if $A = \mathbf{k}Q$ for Q a connected quiver.

Example 3.11. If A is commutative, then $\mathsf{HH}^0(A) = A$, since every element is central.

3.5. **First Hochschild cohomology.** A Hochschild one-cocycle is an element $\phi \in \mathrm{End}_{\mathbf{k}}(A)$ such that $a\phi(b) + \phi(a)b = \phi(ab)$ for all $a, b \in A$. That is these are the derivations of A. A Hochschild one-coboundary is an element $\phi = d(x), x \in A$, and this has the form $\phi(a) = ax - xa$ for all $a \in A$. Therefore, these are the inner derivations. We conclude that

$$\mathsf{HH}^1(A) = \mathrm{Out}(A) := \mathrm{Der}(A)/\,\mathrm{Inn}(A),$$

the vector space of outer derivations of A, which is by definition the quotient of all derivations $\mathrm{Der}(A)$ by the inner derivations $\mathrm{Inn}(A)$. We remark that $\mathrm{Inn}(A) \cong A/Z(A) = A/\mathsf{HH}^0(A)$, since an inner derivation $a \mapsto ax - xa$ is zero if and only if x is central.

Example 3.12. If $A = TV$, then $\mathsf{HH}^1(A) = \mathrm{Out}(TV) = \mathrm{Der}(TV)/\mathrm{Inn}(TV) = \mathrm{Der}(TV)/\overline{TV}$, where $\overline{TV} = TV/\mathbf{k}$, since $Z(TV) = \mathbf{k}$. Explicitly, derivations of TV are uniquely determined by their restrictions to linear maps $V \to TV$, i.e., $\mathrm{Der}(TV) \cong \mathrm{Hom}_{\mathbf{k}}(V, TV)$. So we get $\mathsf{HH}^1(A) \cong \mathrm{Hom}_{\mathbf{k}}(V, TV)/\overline{TV}$.

Example 3.13. If A is commutative, then $\mathrm{Inn}(A) = 0$, so $\mathsf{HH}^1(A) = \mathrm{Der}(A)$. If, moreover, $A = \mathcal{O}_X$ is the commutative algebra of functions on an affine variety, then this is also known as the global vector fields T_X, as we discussed.

Example 3.14. In the case $A = C^\infty(X)$, restricting our Hochschild cochain complex to differential operators in accordance with Remark 3.7, then $\mathsf{HH}^1(A)$ is the space of smooth vector fields on X.

Example 3.15. In the case that V is smooth affine (or more generally normal) and G acts by automorphisms on V and acts freely outside of a codimension-two subset, then all vector fields on V/G lift to the (smooth) locus where G acts freely, so we conclude that $\mathsf{HH}^1(V/G) = \mathrm{Vect}(V/G) = \mathrm{Vect}(V)^G$ is the same as the space of G-invariant vector fields on V. In particular this includes the case where V is a symplectic vector space and $G < \mathsf{Sp}(V)$, since the hyperplanes with nontrivial stabilizer group must be symplectic and hence of codimension at least two.

3.6. **Infinitesimal deformations and second Hochschild cohomology.** We now come to a *key point.* A Hochschild two-cocycle is an element $\gamma \in \mathrm{Hom}_{\mathbf{k}}(A \otimes A, A)$ satisfying

$$a\gamma(b \otimes c) - \gamma(ab \otimes c) + \gamma(a \otimes bc) - \gamma(a \otimes b)c = 0. \tag{3.16}$$

This has a nice interpretation in terms of infinitesimal deformations:

Definition 3.17. Given an augmented commutative ring R with augmentation ideal R_+, a flat deformation of A over R is an R-algebra A' which is isomorphic to $A \otimes_{\mathbf{k}} R$ as an R-module, such that $A' \otimes_R R/R_+ \cong A$ as a $\mathbf{k}$-algebra.

In other words, a deformation of A over R is an algebra $(A \otimes_{\mathbf{k}} R, \star)$ such that $a \star b \equiv ab \pmod{R_+}$.

Definition 3.18. An infinitesimal deformation of A is a flat deformation over $R = \mathbf{k}[\varepsilon]/(\varepsilon^2)$.

Explicitly, an infinitesimal deformation is given by a linear map $\gamma : A \otimes A \to A$, by the formula

$$a \star_\gamma b = ab + \gamma(a \otimes b).$$

Then, the associativity condition is exactly (3.16).

Moreover, we can interpret two-coboundaries as *trivial* infinitesimal deformations. More generally, we say that two infinitesimal deformations γ_1, γ_2 are *equivalent* if there is a $\mathbf{k}[\varepsilon]/(\varepsilon^2)$-module automorphism of A_ε which is the identity modulo ε which takes γ_1 to γ_2. Such a map has the form $phi := \mathrm{Id} + \varepsilon \cdot \phi_1$ for some linear map $\phi_1 : A \to A$, i.e., $\phi_1 \in C^1(A)$. We then compute that

$$\phi^{-1}\big(\phi(a) \star_\gamma \phi(b)\big) = a \star_{\gamma + d\phi_1} b.$$

We conclude

Proposition 3.19. $HH^2(A)$ is the vector space of equivalence classes of infinitesimal deformations of A.

Example 3.20. For $A = TV$, a tensor algebra, we claim that $HH^2(A) = 0$, so there are no nontrivial infinitesimal deformations. Indeed, one can construct a short bimodule resolution of A,

$$0 \to A \otimes V \otimes A \to A \otimes A \twoheadrightarrow A,$$
$$a \otimes v \otimes b \mapsto av \otimes b - a \otimes vb, \quad a \otimes b \mapsto ab.$$

Since this is a projective resolution of length one, we conclude that $\mathrm{Ext}^2_{A^e}(A, M) = 0$ for all bimodules M, i.e., $\mathsf{HH}^2(A, M) = 0$ for all bimodules M.

Example 3.21. If $A = \mathsf{Weyl}(V)$ for a symplectic vector space V, then $\mathsf{HH}^\bullet(A) = \mathbf{k}$, and $\mathsf{HH}^2(\mathsf{Weyl}(V)) = 0$. Moreover, $\mathsf{HH}_\bullet(A) = \mathbf{k}[-\dim V]$ (i.e., $\mathbf{k}$ in degree $\dim V$ and zero elsewhere). To see this, we need a resolution of A as an A-bimodule. This is given by the *Koszul complex*: first consider, for the algebra $\mathsf{Sym}\, V$, the Koszul resolution of the augmentation module $\mathbf{k}$:

$$0 \to \mathsf{Sym}\, V \otimes \wedge^{\dim V} V \to \mathsf{Sym}\, V \otimes \wedge^{\dim V - 1} V \to \cdots \to \mathsf{Sym}\, V \otimes V \to \mathsf{Sym}\, V \twoheadrightarrow \mathbf{k}, \tag{3.22}$$

$$f \otimes (v_1 \wedge \cdots \wedge v_i) \mapsto \sum_{j=1}^{i} (-1)^{j-1} (f v_j) \otimes (v_1 \wedge \cdots \hat{v}_j \cdots \wedge v_i), \tag{3.23}$$

where $\hat{v}_j$ means that v_j was omitted from the wedge product.

Now, this complex deforms to give a resolution of $\mathsf{Weyl}(V)$:

Remark 3.24. The Koszul resolutions above imply that $\mathsf{Weyl}(V)$ and $\mathsf{Sym}\, V$ are *Calabi-Yau algebras of dimension* $\dim V$, as defined in Rogalski's lecture: see Definition 3.27 below. One can also conclude from this that $\mathsf{Weyl}(V) \rtimes G$ and $\mathsf{Sym}\, V \rtimes G$ are CY of dimension $\dim V$; see the next exercise.

Recall from Rogalski's lecture the following. Since we are not assuming A is graded, we will need to replace A^σ with arbitrary invertible bimodules:

Definition 3.25. An A-bimodule U is invertible if there exists an A-bimodule V such that $U \otimes_A V \cong A$ as A-bimodules.

Question 3.26. Is this the same as existing V such that $V \otimes_A U \cong A$ as A-bimodules?

Definition 3.27. A is a Calabi-Yau (CY) algebra of dimension d if A has a projective A-bimodule resolution of length d, and $\mathsf{HH}^\bullet(A, A \otimes A) = A[-d]$ as a graded A-bimodule.

More generally, A is *twisted CY* of dimension d if A has a projective A-bimodule resolution of length d, and $\mathsf{HH}^\bullet(A, A \otimes A) = U[-d]$ as a graded A-bimodule, where U is an invertible A-bimodule.

Here, in $\mathsf{HH}^i(A, A \otimes A)$, $A \otimes A$ is considered as a bimodule with the *outer* bimodule structure, and the remaining inner structure induces a bimodule structure on A; see the next exercise.

Equivalently, A is CY of dimension d if $\mathrm{Ext}^\bullet_{A^e}(A, A^e) = A[-d]$. Expanding the RHS, this says $\mathsf{HH}^i(A, A \otimes A) = 0$ except for $i = d$ where $\mathsf{HH}^d(A, A \otimes A) = A$, as an A-bimodule.

Remark 3.28. In the case A is nonnegatively graded and $A^0 = \mathbf{k}$, then all invertible graded bimodules are of the form A^σ where $\sigma : A \to A$ is a graded automorphism, and the A-bimodule action on A^σ is given by $a \cdot m \cdot b = am\sigma(b)$ for $a, b \in A$ and $m \in A^\sigma$.

Also, recall here that the A-bimodule A^σ is defined to be A as a vector space, with the A-bimodule action $a \cdot m \cdot b = am\sigma(b)$, for $a, b \in A$ and $m \in A^\sigma$, using the usual multiplication in A on the RHS.

Remark 3.29. Note that the condition that A has a projective A-bimodule resolution of length d is equivalent to A having *Hochschild dimension* d, i.e., that $\mathsf{HH}^i(A, M) = 0$ for all $i > d$. It turns out that it is equivalent, in the definition, to only assume A has finite Hochschild dimension, rather than Hochschild dimension d, because under this assumption, the Van den Bergh duality theorem (Theorem 5.3) still holds, and that implies that $\mathsf{HH}^i(A, M) = 0$ for $i > d$ and all bimodules M.

Remark 3.30. In [VdB98], Van den Bergh shows that, if A has Hochschild dimension d and that A is finitely-generated and *bimodule coherent*, i.e., any morphism of finitely generated free A-bimodules has a finitely-generated kernel (note that this is stronger than A being finitely-generated and Noetherian, unlike in the commutative case), then A is automatically twisted Calabi-Yau of dimension d. In this case, $V_A := \mathsf{HH}^d(A, A \otimes A)$ is called the *Van den Bergh dualizing module* of A.

Exercise 3.31. (a) Prove from the Koszul resolution of the example that $\mathsf{Sym}\, V$ and $\mathsf{Weyl}(V)$ are CY of dimension $\dim V$.

(b) Now, take the Koszul resolutions and apply $M \mapsto M \otimes \mathbf{k}[G]$ to all terms, considered as bimodules over $A \rtimes G$ where A is either $\mathsf{Sym}\, V$ or $\mathsf{Weyl}(V)$. This bimodule structure is given by

$$(a \otimes g)(m \otimes h)(a' \otimes g') = (a \cdot g(m) \cdot gh(a')) \otimes (ghg').$$

Prove that the result are resolutions of $A \rtimes G$ as a bimodule over itself. Conclude that $A \rtimes G$ is also CY of dimension $\dim V$.

(c) This item was deleted, as it was false !

(d) Verify that $\mathsf{HH}^\bullet(A, A \otimes A)$ is a canonically an A-bimodule using the *inner* action. Now, verify in part (b) that, in the case of the algebras A and $A \rtimes G$ there, they are indeed CY taking into account this bimodule structure.

(e) In the more general case that $\mathsf{HH}^d(A, A \otimes A) \cong A$ only as an A-*module* (both left and right, but not simultaneously as a bimodule), then A is called *twisted CY*. Prove that the condition that $\mathsf{HH}^d(A, A \otimes A) \cong A$ as both left and right modules, but not necessarily simultaneously, is equivalent to saying that $\mathsf{HH}^d(A, A \otimes A) \cong A^\sigma$, where $\sigma : A \to A$ is an algebra automorphism, and for all $a, b \in A$ and $m \in A^\sigma$, the bimodule action is defined by $a \cdot m \cdot b = (am\sigma(b))$.

3.7. **Obstructions to second-order deformations and third Hochschild cohomology.** Suppose now that we have an infinitesimal deformation given by $\gamma_1 : A \otimes A \to \varepsilon \cdot A$. To extend this to a second-order deformation, we require $\gamma_2 : A \otimes A \to \varepsilon^2 \cdot A$, such that

$$a \star b := ab + \varepsilon\gamma_1(a \otimes b) + \varepsilon^2\gamma_2(a \otimes b)$$

defines an associative product on $A \otimes \mathbf{k}[\varepsilon]/(\varepsilon^2)$.

Looking at the new equation in second degree, this can be written as

$$a\gamma_2(b \otimes c) - \gamma_2(ab \otimes c) + \gamma_2(a \otimes bc) - \gamma_2(a \otimes b)c = \gamma_1(\gamma_1(a \otimes b) \otimes c) - \gamma_1(a \otimes \gamma_1(b \otimes c)).$$

The LHS is $d\gamma_2$, so the condition for γ_2 to exist is exactly that the RHS is a Hochschild coboundary. Moreover, one can easily check that the RHS is a Hochschild two-cocycle (we will give a more conceptual explanation when we discuss the Gerstenhaber bracket). So the element on the RHS defines a class of $\mathsf{HH}^3(A)$ which is the *obstruction* to extending A_ε to a second-order deformation:

Corollary 3.32. $HH^3(A)$ is the space of *obstructions* to extending first-order deformations to second-order deformations. If $HH^3(A) = 0$, then all first-order deformations extend to second-order deformations.

Exercise 3.33. Show, in fact, that the obstruction to extending an n-th order deformation $\sum_{i=1}^n \varepsilon^i \gamma_i$ (where here we have set $\varepsilon^{n+1} = 0$ to a $(n+1)$-st order deformation $\sum_{i=1}^n \varepsilon^i \gamma_i$ (now setting $\varepsilon^{n+2} = 0$, i.e., the existence of a γ_{n+1} so that this defines an associative multiplication on $A \otimes \mathbf{k}[\varepsilon]/(\varepsilon^{n+2})$, is also a class in $HH^3(A)$.

Moreover, if this class vanishes, show that two different choices of γ_{n+1} differ by Hochschild two-cocycles, and that two are equivalent (by applying a $\mathbf{k}[\varepsilon]/(\varepsilon^{n+2})$-module automorphism of $A \otimes \mathbf{k}[\varepsilon]/(\varepsilon^{n+2})$ of the form $\mathrm{Id} + \varepsilon^{n+1} \cdot f$) if and only the two choices of γ_{n+1} differ by a Hochschild two-coboundary. Hence, when the obstruction in $\mathsf{HH}^3(A)$ vanishes, the set of possible extensions to a $(n+1)$-st order deformation form a set isomorphic to $\mathsf{HH}^2(A)$ (more precisely, it forms a *torsor* over the vector space $\mathsf{HH}^2(A)$, i.e., an affine space modeled on $\mathsf{HH}^2(A)$ without a zero element). We will give a more conceptual explanation when we discuss formal deformations.

Note that, when $\mathsf{HH}^3(A) \neq 0$, it can still happen that all infinitesimal deformations extend to all orders. For exqample, by Theorem 2.9, this happens for Poisson structures on smooth manifolds (a Poisson structure yields an infinitesimal deformation by, e.g., $a \star b = ab + \frac{1}{2}\{a, b\} \cdot \varepsilon$; this works for arbitrary skew-symmetric biderivations $\{-, -\}$, but only the Poisson ones, i.e., those satisfying the Jacobi identity, extend to all orders).

However, finding this quantization is *nontrivial*: even though Poisson bivector fields are those classes of $\mathsf{HH}^2(A)$ whose obstruction in $\mathsf{HH}^3(A)$ to extending to second order vanishes, if one does not pick the extension correctly, one *can* obtain an obstruction to continuing to extend to third order, etc. In fact, the proof of Theorem 2.9 describes the space of *all* quantizations: as we will see, deformation quantizations are equivalent to formal deformations of the Poisson structure.

3.8. **Deformations of modules and Hochschild cohomology.** Let A be an associative algebra and M a module over A. Recall that Hochschild (co)homology must take coefficients in an A-*bimodule*, not an A-module. Given M, there is a canonical associated bimodule, namely $\mathrm{End}_{\mathbf{k}}(M)$ (this is an A-bimodule whether M is a left or right module; the same is true for $\mathrm{Hom}_{\mathbf{k}}(M, N)$ where M and N are both left modules, or alternatively both right modules).

Lemma 3.34. $\mathsf{HH}^i(A, \mathrm{End}_{\mathbf{k}}(M)) \cong \mathrm{Ext}^i_A(M, M)$ for all $i \geq 0$. More generally, $\mathsf{HH}^i(A, \mathrm{Hom}_{\mathbf{k}}(M, N)) \cong \mathrm{Ext}^i_A(M, N)$ for all A-modules M and N.

Proof. We prove the second statement. First of all, for $i = 0$,

$$\mathsf{HH}^0(A, \mathrm{Hom}_{\mathbf{k}}(M, N)) = \{\phi \in \mathrm{Hom}_{\mathbf{k}}(M, N) \mid a \cdot \phi = \phi \cdot a, \forall a \in A\} = \mathrm{Hom}_A(M, N).$$

Then the statement for higher i follows because they are the derived functors of the same bifunctors $(A - \mathrm{mod} \times A - \mathrm{mod}) \to \mathbf{k} - mod$.

Explicitly, if $P_\bullet \twoheadrightarrow A$ is a projective A-bimodule resolution of A, then $P_\bullet \otimes_A M \twoheadrightarrow M$ is a projective A-module resolution of M, and

$$\mathrm{RHom}^\bullet_A(M, N) = \mathrm{Hom}_A(P_\bullet \otimes_A M, N)) = \mathrm{Hom}_{A^e}(P_\bullet, \mathrm{Hom}_{\mathbf{k}}(M, N)) = \mathrm{RHom}^\bullet(A, \mathrm{Hom}_{\mathbf{k}}(M, N)),$$

where for the second equality, we used the adjunction $\mathrm{Hom}_B(X \otimes_A Y, Z) = \mathrm{Hom}_{B \otimes A^{\mathrm{op}}}(X, \mathrm{Hom}_{\mathbf{k}}(Y, Z))$, where X is a (B, A)-bimodule, Y is a left A-module, and Z a left B-module. □

In particular, this gives the most natural interpretation of $\mathsf{HH}^0(A, \mathrm{End}_{\mathbf{k}}(M))$: this is just $\mathrm{End}_A(M)$. For the higher groups we recall the following standard descriptions of $\mathrm{Ext}^1(M, M)$ and $\mathrm{Ext}^2(M, M)$, which are convenient to see using Hochschild cochains valued in M.

Definition 3.35. A deformation of an A-module M over an augmented commutative $\mathbf{k}$ algebra $R = \mathbf{k} \oplus R_+$ is an A-module structure on $M \otimes_{\mathbf{k}} R$ such that $(M \otimes_{\mathbf{k}} R) \otimes R/R_+ \cong M$ as an A-module.

Let M be an A-module and let $\rho : A \to \mathrm{End}_{\mathbf{k}}(M)$ be the original (undeformed) module structure.

Proposition 3.36.
(i) The space of Hochschild one-cocycles valued in $\mathrm{End}_{\mathbf{k}}(M)$ is the space of *infinitesimal deformations* of the module M over $R = \mathbf{k}[\varepsilon]/(\varepsilon^2)$;
(ii) Two such deformations are equivalent up to an R-module automorphism of $M \otimes_{\mathbf{k}} R$ which is the identity modulo ε if and only if they differ by a Hochschild one-coboundary.
Thus $\mathsf{HH}^1(A, \mathrm{End}_{\mathbf{k}}(M)) \cong \mathrm{Ext}^1_A(M, M)$ classifies infinitesimal deformations of M.
(iii) The obstruction to extending an infinitesimal deformation with class $\gamma \in HH^1(A, \mathrm{End}_{\mathbf{k}}(M))$ to a second-order deformation, i.e., over $\mathbf{k}[\varepsilon]/(\varepsilon^3)$, is the element

$$\gamma \cup \gamma \in \mathrm{Ext}^2(M, M) \cong \mathsf{HH}^2(A, \mathrm{End}_{\mathbf{k}}(M)),$$

where $\cup$ is the Yoneda cup product of extensions.

For the proof, we recall the following general fact: if N is an A-bimodule, then A-bimodule derivations of N are the same as square-zero algebra extensions $A \oplus N$, i.e., algebra structures $(A \oplus N, \star)$ such that $N \star N = 0$, the bimodule action by $\star$ of A on N is the given one, and $a \star b \equiv ab$ mod N.

Exercise 3.37. Prove the above assertion. Similarly:

(i) If B is a commutative algebra and N a B-module, show that commutative algebra derivations $\mathrm{Der}(B, N)$ are the same as square-zero commutative algebra extensions $B \oplus N$ of B.

(ii) If $\mathfrak{g}$ is a Lie algebra and N a $\mathfrak{g}$-module, show that Lie algebra derivations $\mathrm{Der}(\mathfrak{g}, N)$ are the same as square-zero Lie algebra extensions $\mathfrak{g} \oplus N$ of $\mathfrak{g}$.

Proof of Proposition 3.36. (i) Hochschild one-cocycles are precisely $\gamma \in \mathrm{Hom}(A, \mathrm{End}_{\mathbf{k}}(M))$ such that $\gamma(ab) = a\gamma(b) + \gamma(a)b$, i.e., A-bimodule derivations of $\mathrm{End}_{\mathbf{k}}(M)$. Then the proof follows from the observation after the statement of the proposition.

(ii) If we apply an automorphism $\phi = \mathrm{Id} + \varepsilon \cdot \phi_1$ of $M \otimes_{\mathbf{k}} R$, for $\phi_1 \in \mathrm{End}_{\mathbf{k}}(M)$, then the infinitesimal deformation γ is taken to γ', where

$$(\rho + \varepsilon\gamma')(a) = \phi \circ (\rho + \varepsilon\gamma)(a) \circ \phi^{-1} = (\rho + \varepsilon\gamma)(a) + \varepsilon\phi \circ \rho(a) - \rho(a) \circ \phi = (\rho + \varepsilon(\gamma + d\phi))(a).$$

This proves that $\gamma' - \gamma = d\phi$, as desired. The converse is similar and is left to the reader.

(iii) Working over $\tilde{R} := \mathbf{k}[\varepsilon]/(\varepsilon^3)$, given a Hochschild one-cocycle γ_1, and an arbitrary element $\gamma_2 \in C^2(A, \mathrm{End}_{\mathbf{k}}(M))$,

$$(\rho+\varepsilon\gamma_1+\varepsilon^2\gamma_2)(ab)-(\rho+\varepsilon\gamma_1+\varepsilon^2\gamma_2)(a)\rho+\varepsilon\gamma_1+\varepsilon^2\gamma_2)(b) = \varepsilon^2\big(\gamma_2(ab)-\gamma_1(a)\gamma_1(b)-\gamma_2(a)\rho(b)-\rho(a)\gamma_2(b)\big),$$

and the last expression equals $\varepsilon^2 \cdot (\gamma_1 \cup \gamma_1 + d\gamma_2)(ab)$. Thus the obstruction to extending the module structure is the class $[\gamma_1 \cup \gamma_1] \in \mathsf{HH}^2(A, \mathrm{End}_{\mathbf{k}}(M))$. □

Finally, we can study general deformations:

Definition 3.38. Given an A-module M and a (formal) deformation A_R of A over $R = \mathbf{k} \oplus R_+$, a (formal) deformation of M to an A_R-module is an A_R-module structure on $M \otimes_{\mathbf{k}} R$ or $M \hat{\otimes}_{\mathbf{k}} R$ whose tensor product over R with $R/R_+ = \mathbf{k}$ recovers M.

In the case that A_R is the trivial deformation over R, we also call this a (formal) deformation of the A-module M over R.

Analogously to the above, one can study (uni)versal formal deformations of M; the obstruction to their existence is in $\hbar \cdot \mathsf{HH}^2(A, \mathrm{End}_{\mathbf{k}}(M))[[\hbar]]$, and the base of a universal formal deformation space is a formal affine space over $\mathsf{HH}^1(A, \mathrm{End}_{\mathbf{k}}(M))$.

In more detail, the calculations of Proposition 3.36 generalize to show that, if $\theta \in \hbar \cdot C^2(A, A)[[\hbar]]$ gives a formal deformation $A_\hbar$ of A, then the condition for $\gamma \in \hbar \cdot C^1(A, \mathrm{End}_{\mathbf{k}} M)[[\hbar]]$ to give a formal deformation $M_\hbar$ of M to a module over $A_\hbar$ is

$$\rho \circ \theta = -\gamma \circ \theta + d\gamma + \gamma \cup \gamma, \tag{3.39}$$

where here $(\gamma \cup \gamma)(a \otimes b) := \gamma(a)\gamma(b)$.

Example 3.40. In the presence of a multiparameter formal deformation $(A[[t_1, \ldots, t_n]], \star)$ of A, this can be used to show the existence of a deformation $M_\hbar$ over some restriction of the parameter space. Let $U = \langle t_1, \ldots, t_n \rangle$ and let $\eta : U \to \mathsf{HH}^2(A, A)$ be the map which gives the class of infinitesimal deformation of A. We will need the composition $\rho \circ \eta : U \to \mathsf{HH}^2(A, \mathrm{End}\, M)$. Then one can deduce from the above

Proposition 3.41. (see, e.g., [EM05, Proposition 4.1]) Suppose that the map $\rho \circ \eta$ is surjective with kernel K. Then there exists a formal deformation $(M \otimes \mathcal{O}_S, \rho_S)$ of M over a formal subscheme S of the formal neighborhood of the origin of U, with tangent space K at the origin, which is a module over $(A \otimes \mathcal{O}_S, \star|_S)$. Moreover, if $\mathsf{HH}^1(A, \mathrm{End}\, M) = 0$, then this deformation $M_\hbar$ is unique up to $\mathcal{O}_S$-linear isomorphisms which are the identity modulo $(\mathcal{O}_S)_+$.

In [EM05], this was used to show the existence of a unique family of irreducible representations of a wreath product Cherednik algebra $H_{1,(k,c)}(\Gamma^n \rtimes S_n)$ for $\Gamma < \mathsf{SL}_2(\mathbf{C})$ finite, deforming a module of the form $Y^{\otimes n} \otimes V$ for Y an irreducible module over $H_{1,c_0}(\Gamma)$ and V a particular irreducible representation of S_n (whose Young diagram is a rectangle).

4. Lecture 4: Dglas, the Maurer-Cartan formalism, and proof of formality theorems

As remarked at the beginning of the previous lecture, now the distinction between dg objects and ungraded objects becomes important (especially for the purpose of signs): we will recall in particular the notion of dg Lie algebras (dglas), which have homological grading, and hence parity (even or odd degree).

4.1. **The Gerstenhaber bracket on Hochschild cochains.**

Now, we turn to a promised fundamental structure of Hochschild cochains: the Lie bracket, which is called its *Gerstenhaber bracket*:

Definition 4.1. The *circle product* of Hochschild cochains $\gamma \in C^m(A), \eta \in C^n(A)$ is the element $\gamma \circ \eta \in C^{m+n-1}(A)$ given by

$$\gamma\circ\eta(a_1\otimes\cdots\otimes a_{m+n-1}) := \sum_{i=1}^{m}(-1)^i\gamma(a_1\otimes\cdots\otimes a_{i-1}\otimes\eta(a_i\otimes\cdots\otimes a_{i+n-1})\otimes a_{i+n}\otimes\cdots\otimes a_{m+n-1}). \tag{4.2}$$

Definition 4.3. The *Gerstenhaber bracket* $[\gamma, \eta]$ of $\gamma \in C^m(A), \eta \in C^n(A)$ is

$$[\gamma, \eta] := \gamma \circ \eta - (-1)^{(m+1)(n+1)}\eta \circ \gamma.$$

Definition 4.4. A differential graded (or dg) Lie algebra is a complex $(\mathfrak{g}^\bullet, d)$ together with a bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[x, y] = -(-1)^{|x||y|}[y, x], \quad [x, [y, z]] + (-1)^{|x|(|y|+|z|)}[y, [z, x]] + (-1)^{|z|(|x|+|y|)}[z, [x, y]] = 0, \tag{4.5}$$

$$d[x, y] = [dx, y] + (-1)^{|x|}[x, dy]. \tag{4.6}$$

Remark 4.7. If you are comfortable with the idea of the category of dg vector spaces (i.e., complexes) as equipped with a tensor product (precisely, a symmetric monoidal category), then a dg Lie algebra is exactly a Lie algebra in the category of dg vector spaces.

Definition 4.8. Given a cochain complex C, let $C[m]$ denote the shifted complex, so $(C[m])^i = C^{i+m}$.

In other words, letting C^m denote the ordinary vector space obtained as the degree m part of $C^\bullet$, so C^m by definition is a graded vector space in degree zero, we have

$$C = \bigoplus_{m\in\mathbf{Z}} C^m[-m].$$

Remark 4.9. The circle product also defines a natural structure on $\mathfrak{g} := C(A)[1]$, that of a dg *right pre-Lie* algebra: it satisfies the graded pre-Lie identity

$$\gamma \circ (\eta \circ \theta) - (\gamma \circ \eta) \circ \theta = (-1)^{|\theta||\eta|}\big(\gamma \circ (\theta \circ \eta) - (\gamma \circ \theta) \circ \eta\big),$$

as well as the compatibility with the differential,

$$d(\gamma \circ \eta) = (d\gamma) \circ \eta + (-1)^{|\gamma|}\gamma \circ (d\eta).$$

Given any dg (right) pre-Lie algebra, the obtained bracket

$$[x, y] = x \circ y - (-1)^{|x||y|}y \circ x$$

defines a dg Lie algebra structure.

Exercise 4.10. Verify the assertions of the remark!

The remark (and exercise) immediately imply

Proposition 4.11. The Gerstenhaber bracket defines a dg Lie algebra structure on the shifted complex $\mathfrak{g} := C(A)[1]$.

4.2. **The Maurer-Cartan equation.** We now come to the key description of formal deformations:

Definition 4.12. Let $\mathfrak{g}$ be a dgla. The Maurer-Cartan equation is

$$d\xi + \frac{1}{2}[\xi, \xi] = 0, \quad \xi \in \mathfrak{g}^1. \tag{4.13}$$

A solution of this equation is called a *Maurer-Cartan element.* Denote the space of solutions by $\mathsf{MCE}(\mathfrak{g})$.

The equation can be written suggestively as $[d+\xi, d+\xi] = 0$, if one defines $[d,d] = d^2 = 0$ and $[d,\xi] := d\xi$. In this form the equation is saying that the "connection" $d+\xi$ is flat:

Example 4.14. Here is one of the original instances and motivation of the Maurer-Cartan equation. Let $\mathfrak{g}$ be a Lie algebra and X a manifold or affine algebraic variety X. Then we can consider the dg Lie algebra $(\Omega^\bullet(X) \otimes \mathfrak{g}, d)$, which is the de Rham complex of X tensored with the Lie algebra $\mathfrak{g}$. The grading is given by the de Rham grading, with $|\mathfrak{g}| = 0$. Then, given a $\mathfrak{g}$-valued one-form, α, we can consider $\nabla^\alpha := d + \alpha$, whose curvature is

$$(\nabla^\alpha)^2 = (d+\alpha)^2 = d\alpha + \alpha \wedge \alpha = d\alpha + \frac{1}{2}[\alpha, \alpha]. \tag{4.15}$$

Then the Maurer-Cartan equation for α says that this is zero, i.e., that ∇^α is flat. In other words, Maurer-Cartan element give *deformations of the differential* on the de Rham complex valued in $\mathfrak{g}$ (where α acts via the Lie bracket). In general, this is a good way to think about the Maurer-Cartan equation; see the next remark.

Note that, if G is an algebraic or Lie group such that $\mathfrak{g} = \operatorname{Lie} G$, then ∇^α as above is the same as a *connection* on the trivial principal G-bundle on X, and its curvature is then as defined above.

Closely related to Example 4.14 is the following very important observation.

Proposition 4.16. Suppose $\xi \in \mathsf{MCE}(\mathfrak{g})$.

(i) The map $d^\xi : y \mapsto dy + [\xi, y]$ defines a new differential on $\mathfrak{g}$. Moreover, $(\mathfrak{g}, d^\xi, [-,-])$ is also a dgla.

(ii) Maurer-Cartan elements of $\mathfrak{g}$ are in bijection with those of the twist $\mathfrak{g}^\xi$ by the correspondence

$$\xi + \eta \in \mathfrak{g} \leftrightarrow \eta \in \mathfrak{g}^\xi.$$

Definition 4.17. We call $(\mathfrak{g}, d^\xi, [-,-])$ the *twist by* ξ, and denote it by $\mathfrak{g}^\xi$.

Proof. (i) This is an explicit verification: $(d^\xi)^2(y) = [\xi, dy] + [\xi, [\xi, y]] + d[\xi, y] = [d\xi + \frac{1}{2}[\xi,\xi], y]$, and

$$d^\xi[x,y] - [d^\xi x, y] - (-1)^{|x|}[x, d^\xi y] = [\xi, [x,y]] - [[\xi, x], y] - (-1)^{|x|}[x, [\xi, y]] = 0,$$

where the first equality uses that d is a (graded) derivation for $[-,-]$, and the second equality uses the (graded) Jacobi identity for $[-,-]$.

(ii) One immediately sees that $d^\xi(\eta) + \frac{1}{2}[\eta,\eta] = d(\xi+\eta) + \frac{1}{2}[\xi+\eta, \xi+\eta]$, using that $d\xi + \frac{1}{2}[\xi,\xi] = 0$. □

4.3. **General deformations of algebras.**

Proposition 4.18. One-parameter formal deformations $(A[\![\hbar]\!], \star)$ of an associative algebra A are in bijection with Maurer-Cartan elements of the dgla $\mathfrak{g} := \hbar \cdot (C(A)[1])[\![\hbar]\!]$.

Proof. Let $\gamma := \sum_{m\geq 1} \hbar^m \gamma_m \in \mathfrak{g}^1$. Here $\gamma_m \in C^2(A)$ for all m, since $\mathfrak{g}$ is shifted.

To $\gamma \in \mathfrak{g}^1$ we associate the star product $f \star g = fg + \sum_{m\geq 1} \hbar^m \gamma_m(f \otimes g)$. We need to show that $\star$ is associative if and only if γ satisfies the Maurer-Cartan equation. This follows from a direct computation (see Remark 4.20 for a more conceptual explanation):

$$f \star (g \star h) - (f \star g) \star h = \sum_{m\geq 1} \hbar^m \cdot \big(f\gamma_m(g \otimes h) - \gamma_m(fg \otimes h) + \gamma_m(f \otimes gh) - \gamma_m(f \otimes g)h\big) \tag{4.19}$$
$$+ \sum_{m,n\geq 1} \hbar^{m+n}\big(\gamma_m(f \otimes \gamma_n(g \otimes h)) - \gamma_m(\gamma_n(f \otimes g) \otimes h)\big)$$
$$= d\gamma + \gamma \circ \gamma = d\gamma + \frac{1}{2}[\gamma, \gamma]. \quad \square$$

Remark 4.20. For a more conceptual explanation of the proof, note that, if we let A_0 be an algebra with the zero multiplication, so that $C(A_0, A_0)$ is a dgla with zero differential, then associative multiplications are the same as elements $\mu \in C^2(A_0, A_0) = \mathfrak{g}^1$ satisfying $\frac{1}{2}[\mu, \mu] = 0$, where $\mathfrak{g} = C(A_0, A_0)[1]$ as before. (This is the Maurer-Cartan equation for A_0.) If we set $A_0 := A[\![\hbar]\!]$, and let $\mu \in \mathfrak{g}^1$ be the usual (undeformed) multiplication on A, then for $\gamma := \sum_{m\geq 1} \hbar^m \gamma_m$, the product $\mu + \gamma$ is associative if and only if

$$0 = [\mu + \gamma, \mu + \gamma] = [\mu, \mu] + 2[\mu, \gamma] + [\gamma, \gamma] = 2(d\gamma + \frac{1}{2}[\gamma, \gamma]),$$

where $[\mu, \mu] = 0$ since μ is associative, and $[\mu, \gamma] = d\gamma$ follows immediately from the definition of the Hochschild differential.

More conceptually, the fact $[\mu, \gamma] = d\gamma$ is saying that $C(A) = C(A_0, A_0)^\mu$, the twist of the abelian dgla $C(A)$ by the Maurer-Cartan element μ; cf. Proposition 4.16. In fact, by that remark, we can say: associative multiplications on A_0 are the same as Maurer-Cartan elements ξ, which are the same as Maurer-Cartan elementss $\xi - \mu$ on the twist $A_0^\mu = (A[\![\hbar]\!], \mu)$, by the final comment in Proposition 4.16.

Remark 4.21. The above formalism works, with the same proof, for arbitrary formal and nilpotent deformations, not just one-parameter formal deformations. The general setup is: Let $R = \mathbf{k} \oplus R_+$ be an algebra where R_+ is a pronilpotent ideal. For example, we can have $R = \mathbf{k}[\varepsilon]/(\varepsilon)^2$ or $R = \mathbf{k}[\![t_1, \ldots, t_m]\!]$. Then, associative multiplications on $A \hat{\otimes}_{\mathbf{k}} R$ deforming the associative multiplication μ on A are the same as Maurer-Cartan elements of the pronilpotent dgla $C(A)[1] \otimes_{\mathbf{k}} R_+$.

In the above remark, and below, we will use

Definition 4.22. An ideal I is *pronilpotent* if $\cap_{m\geq 1} I^m = 0$. As an abuse of notation, if $R = \mathbf{k} \oplus R_+$ is an augmented ring, we call R itself pronilpotent if R_+ is.

4.4. **Gauge equivalence.** Recall from Example 4.14 the example of flat connections with values in $\mathfrak{g} = \operatorname{Lie} G$ as solutions of the Maurer-Cartan equation. In that situation, one has a clear notion of equivalence of connections, namely gauge equivalence: for $\gamma : X \to G$ a map, and $\gamma^{-1} : X \to G$ the composition with the inversion on G,

$$\nabla \mapsto (\operatorname{Ad} \gamma)(\nabla); (d + \alpha) \mapsto d + (\operatorname{Ad} \gamma)(\alpha) + \gamma \cdot d(\gamma^{-1}).$$

In the case that $\gamma = \exp(\beta)$ for $\beta \in \mathcal{O}_X \otimes \mathfrak{g}$, we can rewrite this as

$$\alpha \mapsto \exp(\operatorname{ad} \beta)(\alpha) + \frac{1 - \exp(\operatorname{ad} \beta)}{\beta}(d\beta).$$

The last term should be thought of as $\exp(\operatorname{ad}\beta)(d)$, where we set $[d,\beta] = d(\beta)$. See the following exercise to make this rigorous.

Exercise 4.23. Verify this formula: to do so, show that $\gamma \cdot d(\gamma^{-1}) = \sum_{m\geq 0} \frac{1}{m!}(\operatorname{ad}\beta)^{m-1}(-d(\beta))$. Use that $\operatorname{Ad}(\exp(\beta)) = \exp(\operatorname{ad}\beta)$ (this is true for all $\beta \in \mathfrak{g}^0$, and more generally for Lie algebras of connected Lie groups whenever $\exp(\beta)$ makes sense).

Proposition 4.24. Two formal deformations $(A[\![\hbar]\!],\star)$ and $(A[\![\hbar]\!],\star')$ are isomorphic via an automorphism of A which is the identity modulo $\hbar$ if and only if the corresponding Maurer-Cartan elements of $\mathfrak{g} = \hbar \cdot C(A)[\![\hbar]\!]$ are gauge equivalent.

Proof. This is an explicit verification: Let ϕ be an automorphism of $A[\![\hbar]\!]$ which is the identity modulo $\hbar$. We can write $\phi = \exp(\alpha)$ where $\alpha \in \mathfrak{g}^0 = \hbar \operatorname{End}_{\mathbf{k}}(A)[\![\hbar]\!]$, since $\mathfrak{g}$ is pronilpotent (since it has a decreasing filtration by powers of $\hbar$, entirely in degrees ≥ 1). Let $\gamma, \gamma' \in \mathfrak{g}^1$ be the Maurer-Cartan elements corresponding to $\star$ and $\star'$. Let $\mu : A \otimes A \to A$ be the undeformed multiplication. Then

$$\exp(\alpha)(\exp(-\alpha)(a) \star \exp(-\alpha)(b)) = \exp(\operatorname{ad}\alpha)(\mu + \gamma) = \exp(\operatorname{ad}\alpha)(\gamma) + \frac{1-\exp(\operatorname{ad}\alpha)}{\operatorname{ad}\alpha}(d\gamma),$$

where the final equality follows because $[\alpha,\mu] = d\alpha$. □

4.5. **The dgla of polyvector fields, Poisson deformations, and Gerstenhaber algebra structures.** Let X again be a smooth affine algebraic variety or C^∞ manifold. By the Hochschild-Kostant-Rosenberg theorem (Theorem 2.4), the Hochschild cohomology $\mathsf{HH}^\bullet(\mathcal{O}_X)$ is isomorphic to the algebra of polyvector fields, $\wedge^\bullet_{\mathcal{O}_X} T_X$. Since, as we now know, $C^\bullet(\mathcal{O}_X)[1]$ is a dgla, one concludes that $\wedge^\bullet_{\mathcal{O}_X} T_X[1]$ is also a dg Lie algebra (with zero differential). In fact, this structure coincides with the *Schouten-Nijenhuis* bracket, which extends the Lie bracket $[-,-]$ of vector fields:

Proposition 4.25. The Lie bracket on $\wedge^\bullet_{\mathcal{O}_X} T_X[1]$ induced by the Gerstenhaber bracket is the Schouten-Nijenhuis bracket, given by the formula

$$[\xi_1 \wedge \cdots \wedge \xi_m, \eta_1 \wedge \cdots \wedge \eta_n] = \sum_{i,j} (-1)^{i+j+m-1} [\xi_i, \eta_j] \wedge \xi_1 \wedge \cdots \hat{\xi_i} \cdots \wedge \xi_m \wedge \eta_1 \wedge \cdots \hat{\eta}_j \cdots \wedge \eta_n. \tag{4.26}$$

Such a structure is called a *Gerstenhaber algebra*:

Definition 4.27. A (dg) Gerstenhaber algebra is a dg commutative algebra B equipped with a dg Lie algebra structure on the shift $B[1]$, such that (4.26) is satisfied.

Note that, by definition, a Gerstenhaber algebra has to be (homologically) graded; sometimes when the adjective "dg" is omitted one means a dg Gerstenhaber algebra with zero differential. This is the case for $\wedge^\bullet_{\mathcal{O}_X} T_X$.

Remark 4.28. Note that the definition of a Gerstenhaber algebra is very similar to that of a Poisson algebra: the difference is that the Lie bracket on a Gerstenhaber algebra is *odd*: it has homological degree -1.

We easily observe:

Proposition 4.29. A bivector field $\pi \in \wedge^2 T_X$ defines a Poisson bracket if and only if $[\pi,\pi] = 0$. That is, *Poisson bivectors π are solutions of the Maurer-Cartan equation in $\wedge^\bullet_{\mathcal{O}_X} T_X[1]$.*

Exercise 4.30. Prove Proposition 4.29!

The same proof implies:

Corollary 4.31. Formal Poisson structures in $\hbar \cdot \wedge^2_{\mathcal{O}_X} T_X[\![\hbar]\!]$ are the same as Maurer-Cartan elements of the dgla $\hbar \cdot (\wedge^\bullet_{\mathcal{O}_X} T_X[1])[\![\hbar]\!]$.

4.6. **Kontsevich's formality and quantization theorems.** We can now make a precise statement of Kontsevich's formality theorem.

Remark 4.32. Kontsevich proved this result for $\mathbf{R}^n$ or smooth C^∞ manifolds; for the general smooth affine setting, when $\mathbf{k}$ contains $\mathbf{R}$, one can extract this result from [Kon01]; for more details see [Yek05], and also, e.g., [VdB06]. These proofs also yield a sheaf-level version of the statement for the nonaffine algebraic setting. For a simpler proof in the affine algebraic setting, which works over arbitrary fields of characteristic zero, see [DTT07]. That proof uses operadic machinery.

The one parameter version of the theorem is

Theorem 4.33. [Kon03, Kon01, Yek05, DTT07] There is a map

$$\text{Formal Poisson structures on } X \to \text{Formal deformations of } \mathcal{O}_X$$

which induces a bijection modulo continuous automorphisms of $\mathcal{O}_X[\![\hbar]\!]$ which are the identity modulo $\hbar$, and sends a formal Poisson structure $\pi_\hbar$ to a deformation quantization of the ordinary Poisson structure $\pi \equiv \pi_\hbar \pmod{\hbar}$.

The full strength is the following.

Theorem 4.34. [Kon03, Kon01, Yek05, DTT07] There is a map, functorial in dg commutative pronilpotent augmented rings $R = \mathbf{k} \oplus R_+$,

$$\mathcal{U} : \text{Poisson deformations in } \wedge^2_{\mathcal{O}_X} T_X \hat{\otimes}_{\mathbf{k}} R_+ \to \text{Formal deformations } (\mathcal{O}_X \otimes_{\mathbf{k}} R, \star)$$

which induces a bijection modulo continuous automorphisms of $\mathcal{O}_X \hat{\otimes}_{\mathbf{k}} R$. Moreover, modulo R_+^2, this reduces to the identity on bivectors valued in R_+/R_+^2.

Recall here from Definition 4.22 that we call $R = \mathbf{k} \oplus R_+$ pronilpotent when R_+ is pronilpotent. Note that Poisson deformations in $\wedge^2_{\mathcal{O}_X} T_X \hat{\otimes}_{\mathbf{k}} R_+$ are the same as Poisson deformations of the zero Poisson structure on X. When we say "the identity" in the end of the theorem, we note that, working modulo R_+^2, the Jacobi and associativity constraints become trivial. In other words, we are saying that a Poisson deformation gets sent to quantization of the obtained infinitesimal Poisson structure.

4.7. **Twisted version.** Next, we can extend the preceding from deformations of the zero Poisson structure to deformations of a nonzero Poisson structure. Let $\pi_\hbar \in \hbar \cdot \wedge^2_{\mathcal{O}_X} T_X[\![\hbar]\!]$ be a formal Poisson bivector. Thus, $\pi_\hbar$ is a Maurer-Cartan element in the dgla $\hbar \cdot \mathsf{HH}^\bullet(\mathcal{O}_X)[\![\hbar]\!]$.

Twisting $\hbar \cdot \mathsf{HH}^\bullet(\mathcal{O}_X)[\![\hbar]\!]$ by $\pi_\hbar$ and $\hbar \cdot C^\bullet(\mathcal{O}_X)[\![\hbar]\!]$ by the image $\mathcal{U}(\pi_\hbar)$ of $\pi_\hbar$ in $\mathsf{MCE}(D_{\mathrm{poly}}(X))$, we obtain an equivalence, for $(\mathcal{O}_X[\![\hbar]\!], \star)$ the deformation quantization corresponding to $\mathcal{U}(\pi_\hbar)$,

$$\{\text{Formal Poisson deformations of } (\mathcal{O}_X[\![\hbar]\!], \pi_\hbar)\} \leftrightarrow \{\text{Formal deformations of } (\mathcal{O}_X[\![\hbar]\!], \star)\}.$$

Inverting $\hbar$, we obtain

$$\{\text{Formal Poisson deformations of } (\mathcal{O}_X(\!(\hbar)\!), \pi_\hbar)\} \leftrightarrow \{\text{Formal deformations of } (\mathcal{O}_X(\!(\hbar)\!), \star)\}. \tag{4.35}$$

In the case that X is symplectic, the former is given by formal deformations of the symplectic form, i.e., elements of $\hbar \cdot H^2_{DR}(X)[\![\hbar]\!]$, and this proves part of Theorem 2.13, namely, the fact that the given deformation is universal. For the first statement, the isomorphism $H^\bullet_{DR}(X, \mathbf{k}(\!(\hbar)\!)) \xrightarrow{\sim} \mathsf{HH}^\bullet(\mathcal{O}_X(\!(\hbar)\!), \star)$, we use that the dgla controlling Poisson deformations of $(\mathcal{O}_X(\!(\hbar)\!), \pi_\hbar)$ is the dgla computing the Poisson cohomology, and for a symplectic manifold, this is well-known to coincide with the de Rham cohomology.

4.8. **Restatement in terms of morphisms of dglas.** We would like to restate the theorems above without using coefficients in R, just as a statement relating the two dglas in question. Let us name these: $T_{\mathrm{poly}} := \wedge^{\bullet}_{\mathcal{O}_X} T_X[1]$ is the dgla of (shifted) polyvector fields on X, and $D_{\mathrm{poly}} := C^{\bullet}(\mathcal{O}_X)[1]$ is the dgla of (shifted) Hochschild cochains on X, which in the C^∞ setting are required to be differential operators.

These dglas are clearly not isomorphic on the nose, since T_{poly} has zero differential and not D_{poly}. They have isomorphic cohomology, by the Hochschild-Kostant-Rosenberg theorem. In this section we will explain how they are quasi-isomorphic, which is equivalent to the functorial equivalence of Theorem 4.34.

First, the HKR theorem (Theorem 2.4) in fact gives a quasi-isomorphism of complexes $\mathsf{HKR} : T_{\mathrm{poly}} \to D_{\mathrm{poly}}$, defined by

$$\mathsf{HKR}(\xi_1 \wedge \cdots \wedge \xi_m)(f_1 \otimes \cdots \otimes f_m) = \sum_{\sigma \in S_m} \mathrm{sign}(\sigma) \xi_{\sigma(1)}(f_1) \cdots \xi_{\sigma(m)}(f_m).$$

This clearly sends $T^m_{\mathrm{poly}} = \wedge^{m+1}_{\mathcal{O}_X} T_X$ to $D^m_{\mathrm{poly}} = C^{m+1}(\mathcal{O}_X, \mathcal{O}_X)$, since the target is a $\mathcal{O}_X$-multilinear differential operator. Moreover, it is easy to see that the target is closed under the Hochschild differential. By the proof of the HKR theorem, one in fact sees that HKR is a quasi-isomorphism of complexes.

However, HKR is *not* a dgla morphism, since it does not prserve the Lie bracket. It does preserve it when restricted to vector fields, but already does not on bivector fields (which would be needed to apply it in order to take a Poisson bivector field and produce a star product). For example, $[\mathsf{HKR}(\xi_1 \wedge \xi_2), \mathsf{HKR}(\eta_1 \wedge \eta_2)]$ is not, in general, in the image of HKR: it is not skew-symmetric, as one can see by Definition 4.3.

The fundamental idea of Kontsevich was to correct this deficiency by adding higher order terms to HKR. The result will not be a morphism of dglas (this cannot be done), but it will be a more general object called an L_∞ morphism, which we introduce in the next subsection. Such a morphism, from T_{poly} to D_{poly}, is a sequence of linear maps

$$\mathcal{U}_m : \mathsf{Sym}^m(T_{\mathrm{poly}}[1]) \to D_{\mathrm{poly}}[1]$$

satisfying certain axioms. Kontsevich constructs one using graphs as in §2.3, except that now we must allow an arbitrary number of vertices on the real axis, not merely two (the number of vertices corresponds to two more than the degree of the target in $D_{\mathrm{poly}}[1]$), and the outgoing valence of vertices above the real axis can be arbitrary as well. As before, the vertices on the real axis are sinks. Note that $\mathcal{U}_1 = \mathsf{HKR}$ is just the sum of all graphs with a single vertex above the real axis, and all possible numbers of vertices on the real axis.

Then, if we plug in a formal Poisson bivector $\pi_\hbar$, we obtain the star product described in §2.3,

$$f \star g = \sum_{m \geq 1} \frac{1}{m!} \mathcal{U}_m(\pi_\hbar^m)(f \otimes g),$$

i.e., the star product is $\mathcal{U}(\exp(\pi_\hbar))$, where $\mathcal{U} = \sum_{m \geq 1} \mathcal{U}_m$.

4.9. L_∞ **morphisms.** One way to motivate L_∞ morphisms is to study what we require to obtain a functor on Maurer-Cartan elements. We will study this generally for two arbitrary dglas, $\mathfrak{g}$ and $\mathfrak{h}$. First of all, we obviously have:

Proposition 4.36. [Get09] Any dgla morphism $F : \mathfrak{g} \to \mathfrak{h}$ induces a functorial map in augmented pronilpotent dg commutative rings $R = \mathbf{k} \oplus R_+$,

$$F : \mathsf{MCE}(\mathfrak{g} \hat{\otimes}_{\mathbf{k}} R) \to \mathsf{MCE}(\mathfrak{h} \hat{\otimes}_{\mathbf{k}} R).$$

However, it is not true that all functorial maps are obtained from dgla morphisms; in particular, if they were, then they would all extend to functors without requiring pronilpotency (cf. Remark 4.37). We need the pronilpotency, as we can see from the star product formulas of, e.g., §2.3, since otherwise the infinite sum need not converge.

Remark 4.37. In fact, dgla morphisms also induce functorial maps in ordinary dg commutative rings R, taking the ordinary tensor product, without requiring R to be augmented, and with no nilpotency condition. However, the generalization to L_∞-morphisms below requires pronilpotency.

It turns out that there is a complete commutative ring B which represents the functor $R \mapsto \mathsf{MCE}(\mathfrak{g}\hat{\otimes}_{\mathbf{k}}R)$. This means that R-points of $\mathsf{Spf}\, B$, i.e., continuous algebra morphisms $B \to R$, are functorially in bijection with Maurer-Cartan elements of $\mathfrak{g}\hat{\otimes}_{\mathbf{k}}R$. To see what this ring is, first consider the case where $\mathfrak{g}$ is abelian with zero differential. Then Maurer-Cartan elements are merely elements of $\mathfrak{g}^1$. So we want to consider the completed symmetric algebra $\hat{S}(\mathfrak{g}[1])^*$, since $\mathfrak{g}[1]$ is the graded vector space such that $(\mathfrak{g}[1])^0 = \mathfrak{g}^1$. By analogy, given any ungraded vector space V, the algebra of functions $\mathcal{O}_V$ is nothing but $\mathsf{Sym}\, V^*$, so by analogy one should take the same thing when V is graded, where the ordinary points are now V^0. By analogy the same should hold if V is a dg vector space, where now the ordinary points of $\mathsf{Sym}\, V^*$ consist of the zeroth cohomology, $H^0(V)$.

Thus, we consider the completed symmetric algebra $\hat{S}(\mathfrak{g}[1])^*$. In the case $\mathfrak{g}$ is nonabelian, we can account for the Lie bracket by deforming the differential on $\hat{S}(\mathfrak{g}[1])^*$, so that the spectrum consists of Maurer-Cartan elements rather than all of $H^1(\mathfrak{g})$.

The result is the *Chevalley-Eilenberg complex of* $\mathfrak{g}$, which you may already know as the complex computing the Lie algebra cohomology of $\mathfrak{g}$.

Remark 4.38. We need to consider here $(\mathfrak{g}[1])^*$ as the *topological* dual to $\mathfrak{g}[1]$. Since $\mathfrak{g}$ is considered as discrete, this is the same as the ordinary dual equipped with a not-necessarily discrete topology, given by the inverse limit of the duals of the finite-dimensional subspaces of $\mathfrak{g}[1]$,

$$\mathfrak{g}[1]^* := \lim_{\substack{\leftarrow \\ V \subseteq \mathfrak{g}[1] \text{ f.d.}}} V^*.$$

This is an inverse limit of finite-dimensional vector spaces.[1]

Definition 4.39. The Chevalley-Eilenberg complex is the complete dg commutative algebra $C_{CE}(\mathfrak{g}) := (\hat{S}(\mathfrak{g}[1])^*, d_{CE})$, where d is the derivation such that $d_{CE}(x) = d^*_{\mathfrak{g}}(x) + \frac{1}{2}\delta_{\mathfrak{g}}(x)$, where the degree one map $\delta_{\mathfrak{g}} : \mathfrak{g}[1]^* \to \mathsf{Sym}^2\, \mathfrak{g}[1]^*$ is the dual of the Lie bracket $\wedge^2\mathfrak{g} \to \mathfrak{g}$.

Proposition 4.40. Let $R = \mathbf{k} \oplus R_+$ be a dg commutative pronilpotent augmented ring. Then there is a canonical bijection between continuous dg commutative algebra morphisms $C_{CE}(\mathfrak{g}) \to R$ and Maurer-Cartan elements of $\mathfrak{g} \otimes R$, given by restricting to $\mathfrak{g}[1]^*$.

Proof. It is clear that, if we do not consider the differential, continuous commutative algebra morphisms $C_{CE}(\mathfrak{g}) \to R$ are in bijection with continuous graded maps $\chi : \mathfrak{g}[1]^* \to R$. Such elements, because of the continuity requirement, are the same as elements $x_\chi \in \mathfrak{g}[1]\hat{\otimes}R$. Then, χ commutes with the differential if and only if $\chi \circ d^*_{\mathfrak{g}} + \frac{1}{2}(\chi\otimes\chi)\circ\delta_{\mathfrak{g}} = 0$, i.e., if and only if $d(x_\chi) + \frac{1}{2}[x_\chi, x_\chi] = 0$. □

Corollary 4.41. There is a canonical bijection

$$(4.42)\quad \{\text{Functorial in } R \text{ maps } F : MCE(\mathfrak{g}\hat{\otimes}R) \to \mathsf{MCE}(\mathfrak{h}\hat{\otimes}R)\}$$
$$\leftrightarrow \{\text{dg commutative morphisms } F^* : C_{CE}(\mathfrak{h}) \to C_{CE}(\mathfrak{g})\},$$

where R ranges over dg commutative augmented pronilpotent $\mathbf{k}$-algebras.

[1] Such a vector space is often called a *formal vector space*, not to be confused, however, with the notion of formality we are discussing as in Kontsevich's theorem! To avoid confusion, I will not use this term.

Proof. This is a Yoneda type result: given a dg commutative morphism $C_{CE}(\mathfrak{h}) \to C_{CE}(\mathfrak{g})$, the pullback defines a map $MCE(\mathfrak{g} \otimes R) \to \mathsf{MCE}(\mathfrak{h} \otimes R)$ for every R as described, which is functorial in R. Conversely, given the functorial map $\mathsf{MCE}(\mathfrak{g} \otimes R) \to \mathsf{MCE}(\mathfrak{h} \otimes R)$, we apply it to $R = C_{CE}(\mathfrak{g})$ itself. Then, by Proposition 4.40, the identity map $C_{CE}(\mathfrak{g}) \to R$ yields a Maurer-Cartan element $I \in \mathfrak{g}\hat{\otimes} C_{CE}(\mathfrak{g})$ (the "universal" Maurer-Cartan element). Its image in $\mathsf{MCE}(\mathfrak{h}\hat{\otimes} C_{CE}(\mathfrak{g}))$ yields, by Proposition 4.40, a dg commutative morphism $C_{CE}(\mathfrak{h}) \to C_{CE}(\mathfrak{g})$. It is straightforward to check that these maps are inverse to each other. □

Definition 4.43. An L_∞-morphism $F : \mathfrak{g} \to \mathfrak{h}$ is a dg commutative morphism $F^* : C_{CE}(\mathfrak{h}) \to C_{CE}(\mathfrak{g})$.

We will refer to F^* as the *pullback* of F. Thus, the corollary can be alternatively stated as

$$\{\text{Functorial in } R \text{ maps } F : \mathsf{MCE}(\mathfrak{g}\hat{\otimes} R) \to \mathsf{MCE}(\mathfrak{h}\hat{\otimes} R)\} \leftrightarrow \{L_\infty\text{-morphisms } F : \mathfrak{g} \to \mathfrak{h}\}. \quad (4.44)$$

Finally, the above extends to describe quasi-isomorphisms:

Definition 4.45. An L_∞ quasi-isomorphism is a L_∞ morphism which is an isomorphism on homology, i.e., a dg commutative quasi-isomorphism $F^* : C_{CE}(\mathfrak{h}) \to C_{CE}(\mathfrak{g})$.

This can also be called a homotopy equivalence of dglas.

Proposition 4.46. There is a canonical bijection

$$\{\text{Functorial in } R \text{ maps } \mathsf{MCE}(\mathfrak{g}\hat{\otimes} R) \to \mathsf{MCE}(\mathfrak{h}\hat{\otimes} R), \text{ which are bijective modulo gauge equivalence}\} \leftrightarrow \{L_\infty \text{ quasi-isomorphisms } \mathfrak{g} \to \mathfrak{h}\}. \quad (4.47)$$

We omit the proof.

4.10. **Explicit definition of L_∞ morphisms.** Let us write out explicitly what it means to be an L_∞ morphism. Let $\mathfrak{g}$ and $\mathfrak{h}$ be dglas. Then an L_∞ morphism is a cdga (commutative dg algebra) morphism $F^* : C_{CE}(\mathfrak{h}) \to C_{CE}(\mathfrak{g})$. Since $C_{CE}(\mathfrak{h})$ is the symmetric algebra on $\mathfrak{h}[1]^*$, this map is uniquely determined by its restriction to $\mathfrak{h}[1]^*$. We then obtain a sequence of maps

$$F_m^* : \mathfrak{h}[1]^* \to \mathsf{Sym}^m \mathfrak{g}[1]^*,$$

or dually,

$$F_m : \mathsf{Sym}^m \mathfrak{g}[1] \to \mathfrak{h}[1].$$

The F_m are the Taylor coefficients of F, since they are the parts of F^* of polynomial degree m, i.e., the order-m Taylor coefficients of the map on Maurer-Cartan elements.

The condition that F^* commute with the differential says that, if

$$F^*(d_{\mathfrak{h}}^*(x)) + F^*(\frac{1}{2}\delta_{\mathfrak{h}}(x)) = d_{\mathfrak{g}}^* F^*(x) + \frac{1}{2}\delta_{\mathfrak{g}} F^*(x).$$

Writing this in terms of the F_m, we obtain that, for all $m \geq 1$,

$$d_{\mathfrak{h}} \circ F_m + \frac{1}{2}\sum_{i+j=m} [F_i, F_j]_{\mathfrak{h}} = F_m \circ d_{CE,\mathfrak{g}}, \quad (4.48)$$

where $d_{CE,\mathfrak{g}}$ is the Chevalley-Eilenberg differential for $\mathfrak{g}$.

Now, let us specialize to $\mathfrak{g} = T_{\text{poly}}$ and $\mathfrak{h} = D_{\text{poly}}$, with $F_m = \mathcal{U}_m$ for all $m \geq 1$. In terms of Kontsevich's graphs, the second term of (4.48) involves a sum over all ways of combining two graphs together by a single edge to get a larger graph, multiplying the weights for those graphs. The last term (the RHS) of (4.48) involves summing over all ways of expanding a graph by adding a single edge. The first term on the LHS says to apply the Hochschild differential to the result of all graphs, and this can be suppressed in exchange for adding to Kontsevich's map $\mathcal{U}$ a term $\mathcal{U} : \mathbf{R} = \mathsf{Sym}^0 T_{\text{poly}}[1] \to D_{\text{poly}}[1]$ which sends $1 \in \mathbf{R}$ to $\mu_A \in C^2(A, A)$. (Recall that $\mathbf{k} = \mathbf{R}$ for Kontsevich's construction; this is needed to define the weights of the graphs.)

4.11. **Formality in terms of a L_∞ quasi-isomorphism.** We deduce from the preceding material that the formality theorem, Theorem 4.34, can be restated as

Theorem 4.49. [Kon03, Kon01, Yek05, DTT07] There is an L_∞ quasi-isomorphism,

$$T_{\text{poly}}(X) \xrightarrow{\sim} D_{\text{poly}}(X).$$

The proof is accomplished in [Kon03] by finding weights w_Γ to attach to all of the graphs Γ described above, so that the explicit equations of the preceding section are satisfied. These explicit equations are quadratic in in the weights, and are of the form, for certain graphs Γ,

$$\sum_{\Gamma = \Gamma_1 \cup \{e\} \cup \Gamma_2} w_{\Gamma_1} w_{\Gamma_2} + \sum_{\Gamma = e \cup \Gamma'} w_{\Gamma'}.$$

This is done by defining the w_Γ as certain integrals over partially compactified configuration spaces of vertices of the graph, such that the above sum follows from Stokes' theorem for the configuration space C_Γ associated to Γ, whose boundary strata are of the form $C_{\Gamma_1 \times \Gamma_2}$ or $C_{\Gamma'}$.

Remark 4.50. In fact, T_{poly} and D_{poly} are quasi-isomorphic not merely as L_∞ algebras but in fact as homotopy Gerstenhaber algebras, i.e., including the structure of cup product (and additional "brace algebra" structures in the case of D_{poly}). In [Kon03] Kontsevich shows that his L_∞ morphism is compatible with cup products; in [DTT07] the main result actually constructs a homotopy Gerstenhaber equivalence between $T_{\text{poly}}(X)$ and $D_{\text{poly}}(X)$, in the general smooth affine algebraic setting over characteristic zero.

4.12. **Twisting the L_∞-morphism; Poisson and Hochschild cohomology.** Parallel to Theorem 4.35, given a formal Poisson structure $\pi_\hbar \in \mathsf{MCE}(\hbar \cdot T_{\text{poly}}(X)[\![\hbar]\!])$ and its image star product $\star$, corresponding to the element $\mathcal{U}(\pi_\hbar) \in \mathsf{MCE}(\hbar \cdot D_{\text{poly}}(X)[\![\hbar]\!])$, we obtain a quasi-isomorphism of twisted dglas,

$$T_{\text{poly}}(X)(\!(\hbar)\!)^{\pi_\hbar} \xrightarrow{\sim} D_{\text{poly}}(X)(\!(\hbar)\!)^{\mathcal{U}(\pi_\hbar)}. \tag{4.51}$$

The former computes the Poisson cohomology of $(\mathcal{O}_X(\!(\hbar)\!), \pi_\hbar)$, and the latter computes the Hochschild cohomology of $(\mathcal{O}_X(\!(\hbar)\!), \star)$. We obtain

Corollary 4.52. There are isomorphisms of graded vector spaces,

$$\mathsf{HP}^\bullet(\mathcal{O}_X(\!(\hbar)\!), \pi_\hbar) \xrightarrow{\sim} \mathsf{HH}^\bullet(\mathcal{O}_X(\!(\hbar)\!), \star). \tag{4.53}$$

In particular, applied to degrees $0, 1$, and 2, there are canonical $\mathbf{k}(\!(\hbar)\!)$-linear isomorphisms:

(i) From the Poisson center of $(\mathcal{O}_X(\!(\hbar)\!), \pi_\hbar)$ to the center of $(\mathcal{O}_X(\!(\hbar)\!), \star)$;
(ii) From the outer derivations of $(\mathcal{O}_X(\!(\hbar)\!), \pi_\hbar)$ to the outer derivations of $(\mathcal{O}_X(\!(\hbar)\!), \star)$;
(iii) From infinitesimal Poisson deformations of $(\mathcal{O}_X(\!(\hbar)\!), \star)$ to infinitesimal algebra deformations of $(\mathcal{O}_X(\!(\hbar)\!), \star)$.

Remark 4.54. In fact, the above isomorphism (4.53) is an isomorphism of $\mathbf{k}(\!(\hbar)\!)$-*algebras*, and hence also the morphism (i) is an isomorphism of algebras, and the morphisms (ii)–(iii) are compatible with the module structures over the Poisson center on the LHS and the center of the quantization on the RHS. This is highly nontrivial and was proved by Kontsevich in [Kon03, §8]. More conceptually, the reason why this holds is that, as demonstrated by Willwacher in [Wil11], Kontsevich's L_∞ quasi-isomorphism $\mathsf{HH}^\bullet(\mathcal{O}_X) \xrightarrow{\sim} C^\bullet(\mathcal{O}_X)$ actually can be lifted to a G_∞ morphism (and hence quasi-isomorphism), where G_∞ is the Gerstenhaber algebra version of L_∞, i.e., which takes into account cup products.

In fact, in [Wil11], Willwacher shows that Kontsevich's morphism lifts to a "KS_∞ quasi-isomorphism" of pairs $(\mathsf{HH}^\bullet(\mathcal{O}_X), \mathsf{HH}_\bullet(\mathcal{O}_X)) \xrightarrow{\sim} (C^\bullet(\mathcal{O}_X), C_\bullet(\mathcal{O}_X))$, where we equip the Hochschild homology with the natural operations by the contraction and Lie derivative operations from Hochschild cohomology (i.e., the calculus structure), and similarly equip Hochschild chains with the analogous natural operations by Hochschild cochains.

4.13. **Explicit twisting of L_∞ morphisms.** We caution that, unlike in the untwisted case, (4.53) is **not** obtained merely from the HKR morphism. Indeed, even in degree zero, an element which is Poisson central for $\pi_\hbar$ need not correspond in any obvious way to an element which is central in the quantized algebra. The only obvious statement one could make is that we have a map modulo $\hbar$, where $\pi_\hbar = \hbar \cdot \pi +$ higher order terms, with π an ordinary Poisson structure,

$$Z(\mathcal{O}_X, \pi) \leftarrow Z(\mathcal{O}_X[\![\hbar]\!], \star)/(\hbar)$$

which is quite different (and weaker) than the above statement.

To write the correct formula for the isomorphism (4.53), we need to discuss a more general question we have not addressed: given an L_∞ morphism $F : \mathfrak{g} \to \mathfrak{h}$, what is the induced formula on cohomology? In particular, we need to also compute this for $\mathfrak{g}^\xi \to \mathfrak{h}^{F(\xi)}$, for all $\xi \in \mathsf{MCE}(\mathfrak{g})$.

In terms of the morphism $F^* : C_{CE}(\mathfrak{h}) \to C_{CE}(\mathfrak{g})$, the induced map $H^\bullet(\mathfrak{g}) \to H^\bullet(\mathfrak{h})$ is indeed obtained by composing F^* with the obvious projection $C_{CE}(\mathfrak{g}) = \hat{S}(\mathfrak{g}^*[1]) \twoheadrightarrow \mathfrak{g}^*[1]$. For Kontsevich's morphism $\mathcal{U}$, which is explicitly defined by $\mathcal{U}^*$, this produces the HKR isomorphism $H^\bullet(T_{\mathrm{poly}}(X)) = \wedge^{\bullet+1}_{\mathcal{O}_X} T_X \to H^\bullet(D_{\mathrm{poly}}(X)) \cong \mathsf{HH}^{\bullet+1}(\mathcal{O}_X)$. However, to apply this to the twisted versions $\mathcal{U}^{\pi_\hbar}$: $T_{\mathrm{poly}}(X)^{\pi_\hbar} \to D_{\mathrm{poly}}(X)^{\mathcal{U}(\pi_\hbar)}$, we need an explicit formula for the pullback $(F^\xi)^*$. This is nontrivial by the above observation—on cohomology one does *not* obtain the HKR morphism.

By Proposition 4.16, on Maurer-Cartan elements, given $F : \mathfrak{g} \to \mathfrak{h}$ and $\xi \in \mathsf{MCE}(\mathfrak{g})$, then $F^\xi(\eta) = F(\eta + \xi) - F(\xi)$. Thus, $(F^\xi)^* = (\eta \mapsto \eta + \xi)^* \circ F^* \circ (\theta \mapsto \theta - F(\xi))^*$. Explicitly,

$$(\eta \mapsto \eta + \xi)^*(x) = x + x(\xi), \quad x \in \mathfrak{g}^*[1],$$

and this extends to a morphism of commutative algebras, which is compatible with the differential since $d(x) = d(x + x(\xi))$, as $x(\xi)$ is a constant (note that d is a derivation, so the whole morphism must preserve d when we check it on $\mathfrak{g}^*[1]$).

Explicitly, the formula we obtain on Taylor coefficients F^ξ_m is, for $\eta_1, \ldots, \eta_m \in \mathfrak{g}$,

$$F^\xi_m(\eta_1 \wedge \cdots \wedge \eta_m) = \sum_{k \geq 0} F^\xi_{m+k}(\xi^{\wedge k} \wedge \eta_1 \wedge \cdots \wedge \eta_m). \tag{4.55}$$

Thus, the composition of $(F^\xi)^*$ with the projection yields the map on cohomology,

$$H^\bullet(\mathfrak{g}^\xi) \to H^\bullet(\mathfrak{h}^{F(\xi)}), \quad x \mapsto \sum_{m \geq 1} F_m(x \wedge \xi^{\wedge(m-1)}). \tag{4.56}$$

Let us now explain a geometric interpretation, which is also an alternative proof, which was used by Kontsevich in [Kon03, §8]. Namely, he observes that the cohomology of the twisted dgla $\mathfrak{g}^\xi$ is the *tangent space* in the moduli space $\mathsf{MCE}(\mathfrak{g})/\sim$ to the Maurer-Cartan element ξ. Here $\sim$ denotes gauge equivalence. This is because, if we fix ξ, and differentiate the Maurer-Cartan equation $d(\xi + \eta) + \frac{1}{2}[\xi + \eta, \xi + \eta]$ with respect to η, then the tangent space is $(\mathfrak{g}^1, d + \mathrm{ad}\,\xi)$, and two tangent vectors η and η' are gauge equivalent if and only if they differ by $(d + \mathrm{ad}\,\xi)(z)$ for $z \in \mathfrak{g}^0$. Functorially, this says that the entire dgla $(\mathfrak{g}, d + \mathrm{ad}\,\xi)$ is the dg tangent space to the moduli space of Maurer-Cartan elements of $\mathfrak{g}$, taken with coefficients in arbitrary pronilpotent augmented dg commutative algebras: this is because, if you use such a dg commutative algebra R which is not concentrated in degree zero, then $\mathsf{MCE}(\mathfrak{g} \times R)$ will detect cohomology of $\mathfrak{g}$ which is not merely in degree one.

Therefore, we will adopt Kontsevich's terminology and refer to (4.56) as the *tangent map*, and denote it by $dF|_\xi$ (Kontsevich denotes it by I_T, at least in the situation of $X = \mathfrak{g}^*$ and ξ equal to the $\hbar$ times the standard Poisson structure).

4.14. **The algebra isomorphism** $(\mathsf{Sym}\,\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} Z(U\mathfrak{g})$ **and Duflo's isomorphism.** In the case $X = \mathfrak{g}^*$ and $\mathcal{O}_X = \mathsf{Sym}\,\mathfrak{g}$ for $\mathfrak{g}$ a finite-dimensional Lie algebra, the Poisson center $\mathsf{HP}^0(\mathcal{O}_X) = Z(\mathcal{O}_X)$ is equal to $(\mathsf{Sym}\,\mathfrak{g})^{\mathfrak{g}}$. By Remark 4.54, Kontsevich's morphism induces an isomorphism of algebras

$$(\mathsf{Sym}\,\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} Z(\mathsf{Sym}\,\mathfrak{g}, \star). \tag{4.57}$$

Moreover, by Remark 2.12 and Exercise 1 of Exercise Sheet 4, $(\mathsf{Sym}\,\mathfrak{g}, \star) \cong U_\hbar\mathfrak{g}$, so we obtain from the above an isomorphism

$$(\mathsf{Sym}\,\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} Z(U_\hbar\mathfrak{g}). \tag{4.58}$$

That such an isomorphism exists, for *arbitrary* finite-dimensional $\mathfrak{g}$, is a significant generalization of the Harish-Chandra isomorphism for the semisimple case, which was first noticed by Kirillov and then proved by Duflo, using a highly nontrivial formula. By partially computing his isomorphism (4.57), Kontsevich was able to show that his isomorphism (4.58) coincides with Duflo's isomorphism.

In more detail, as explained in the previous subsection, (4.57) is a *nontrivial* map, given by

$$f \mapsto f + \sum_{m \geq 1} \hbar^m \mathcal{U}_{m+1}(f \wedge \pi^{\wedge m}).$$

In turn, this re-expresses as a sum over all graphs in the upper half plane $\{(x,y) \mid y \geq 0\} \subseteq \mathbf{R}^2$, up to isomorphism, which have a single vertex on the x-axis (corresponding to f), which is a sink, and have m vertices in the upper-half plane, corresponding to π, each of which has two outgoing edges. Moreover, we can discard all the graphs where the two outgoing edges of a given vertex labeled by π have the same target, i.e., we can assume the graph has no *multiple edges*, since π is skew-symmetric, and thus the resulting bilinear operation (cf. §2.3) would be zero.

The remaining graphs thus consist of oriented circles of vertices labeled by π in the upper-half plane, together with another arrow from each vertex labeled by π to the vertex labeled by f on the x-axis. Kontsevich calls such graphs *wheels*, because one can redraw it, forgetting about the condition that the graph lie in the upper half plane, with the vertex labeled by f in the center, and one obtains a wheel.

Kontsevich computes that the weights of such graphs, with k wheels, are the *product* of the weights of each wheel individually, considered as a subgraph. Moreover, by a symmetry argument he uses also elsewhere, he concludes that the weight is zero when k is odd. This is enough information to deduce that, if Kontsevich's morphism (4.58) differed from Duflo's, then it would have to differ by an automorphism of $U_\hbar\mathfrak{g}$ of a form which is impossible, and hence the two coincide.

5. Lecture 5: Calabi-Yau algebras and isolated hypersurface singularities

5.1. **Motivation: quantization of isolated hypersurface singularities and del Pezzo surfaces.** We have seen above that, in many cases, it is much easier to describe deformations of algebras via generators and relations than via star products. For example, this was the case for the Weyl and universal enveloping algebras, as well as the symplectic reflection algebras, where it is rather simple to write down the deformed relations, but the star product is complicated. More generally, given a Koszul algebra, one can study the deformations of the relations that satisfy the PBW property, i.e., give a flat deformation; in nice cases these will yield all deformations, even though writing down the corresponding star products could be difficult.

In our running example of $\mathbf{C}[x,y]^{\mathbf{Z}/2} = \mathbf{C}[x^2,xy,y^2] = \mathcal{O}_{\mathrm{Nil}\,\mathfrak{sl}_2}$, one way we did this was by directly finding a noncommutative deformation of the singular ring $\mathbf{C}[x,y]^{\mathbf{Z}/2}$ itself, but rather by deforming either $\mathbf{C}[x,y]$ to $\mathsf{Weyl}(\mathbf{k}^2)$ and taking $\mathbf{Z}/2$ invariants, or more generally deforming $\mathbf{C}[x,y] \rtimes \mathbf{Z}/2$ to an SRA $H_{1,c}(\mathbf{Z}/2)$ and then passing to the spherical subalgebra $eH_{1,c}(\mathbf{Z}/2)e$.

Another way we did it was by realizing the ring as $\mathcal{O}_{\mathrm{Nil}\,\mathfrak{sl}_2}$ for the subvariety $\mathrm{Nil}\,\mathfrak{sl}_2 \subseteq \mathfrak{sl}_2 \cong \mathbf{A}^3$ inside affine space, then first deforming $\mathcal{O}_{\mathfrak{sl}_2} = \mathcal{O}_{\mathfrak{sl}_2^*}$ to the noncommutative ring $U\mathfrak{sl}_2$, then finding a central quotient that yields a quantization of $\mathcal{O}_{\mathrm{Nil}\,\mathfrak{sl}_2}$.

We would like to generalize this approach to quantizing more general hypersurfaces in $\mathbf{A}^3$. Suppose $f \in \mathbf{k}[x,y,z]$ is a hypersurface. Then we claim that there is a canonical Poisson bivector field on $Z(f)$. This comes from the *Calabi-Yau* structure on $\mathbf{A}^3$, i.e., *everywhere nonvanishing volume form.* Namely, $\mathbf{A}^3$ is equipped with the volume form

$$dx \wedge dy \wedge dz.$$

The inverse of this is the everywhere nonvanishing top polyvector field

$$\partial_x \wedge \partial_y \wedge \partial_z.$$

Now, we can contract this with df and obtain a bivector field,

$$\pi_f := (\partial_x \wedge \partial_y \wedge \partial_z)(df) = \partial_x(f)\partial_y \wedge \partial_z + \partial_y(f)\partial_z \wedge \partial_x + \partial_z(f)\partial_x \wedge \partial_y. \tag{5.1}$$

Exercise 5.2. Show that this is Poisson. Show also that f is Poisson central, so that the quotient $\mathcal{O}_{Z(f)} = \mathcal{O}_{\mathbf{A}^3}/(f)$ is Poisson, i.e., the surface $Z(f) \subseteq \mathbf{A}^3$ is canonically equipped with a Poisson bivector from the Calabi-Yau structure on $\mathbf{A}^3$.

Moreover, show that the Poisson bivector π_f is *unimodular*: for every Hamiltonian vector field $\xi_g := \pi(dg) = \{g, -\}$ with respect to this Poisson bivector, we have $L_{\xi_g}(\mathsf{vol}) = 0$, where $\mathsf{vol} = dx \wedge dy \wedge dz$. Equivalently, $L_\pi(\mathsf{vol}) = 0$.

More generally (but harder), show that, for an arbitrary complete intersection surface $Z(f_1, \ldots, f_{n-2})$ in $\mathbf{A}^n$, then

$$(\partial_1 \wedge \cdots \wedge \partial_n)(df_1 \wedge \cdots \wedge df_{n-2})$$

is a Poisson structure. Show that $f_1, \ldots, f_{n-2}$ are Poisson central, so that the surface $Z(f_1, \ldots, f_{n-2})$ is a closed Poisson subvariety, and in particular has a canonical Poisson structure. Moreover, show that π is *unimodular*: for every Hamiltonian vector field $\xi_g := \pi(dg)$, we have $L_{\xi_g}(\mathsf{vol}) = 0$, where $\mathsf{vol} = dx_1 \wedge \cdots \wedge dx_n$. Equivalently, $L_\pi(\mathsf{vol}) = 0$.

From this, we can deduce that the same holds if $\mathbf{A}^n$ is replaced by an arbitrary n-dimensional Calabi-Yau variety X, i.e., a variety X equipped with a nonvanishing volume form, since the Jacobi condition $[\pi, \pi] = 0$ can be checked in the formal neighborhood of a point of X, which is isomorphic to the formal neighborhood of the origin in $\mathbf{A}^n$.

Now, our strategy, following [EG10], for deforming $X = Z(f) \subseteq \mathbf{A}^3$ is as follows:

(1) First, consider *Calabi-Yau* deformations of $\mathbf{A}^3$, i.e., noncommutative deformations of $\mathbf{A}^3$ as a Calabi-Yau algebra. We should consider these in the direction of the Poisson structure π_f defined above.
(2) Next, inside such a Calabi-Yau deformation A of $\mathcal{O}_{\mathbf{A}^3}$, we identify a central (or more generally normal) element Φ corresponding to $f \in \mathcal{O}_{\mathbf{A}^3}$.
(3) Then, the quantization of $\mathcal{O}_X$ is $A/(\Phi)$.

In order to carry out this program, we need to recall the notion of *(noncommutative) Calabi-Yau algebras* and their convenient presentation by relations derived from a single *potential.* Then it turns out that deforming in the direction of π_f is obtained by deforming the potential of $\mathbf{A}^3$ in the "direction of f." Since f is Poisson central, by Kontsevich's theorem (Corollary 4.52), f deforms to a central element of the quantization.

5.2. **Calabi-Yau algebras and the course.** Recall Definition 3.27 from Rogalski's lecture. These algebras are ubiquitous and in fact they have appeared throughout the course:

(1) A commutative Calabi-Yau algebra is the algebra of functions on a Calabi-Yau affine algebraic variety, i.e., a smooth affine algebraic variety equipped with an everywhere nonvanishing volume form;
(2) Most deformations we have considered of Calabi-Yau algebras are still Calabi-Yau. This includes:
   (a) The universal enveloping algebra $U\mathfrak{g}$ of a finite-dimensional Lie algebra $\mathfrak{g}$ (this is Calabi-Yau of dimension $\dim \mathfrak{g}$);
   (b) The Weyl algebras $\mathsf{Weyl}(V)$ (CY of dimension $\dim V$), as well as their invariant subrings $\mathsf{Weyl}(V)^G$ for $G < \mathsf{Sp}(V)$ finite;
   (c) The skew-group ring $\mathcal{O}_V \rtimes G$ for a vector space V and $G < \mathsf{SL}(V)$ finite;
   (d) All symplectic reflection algebras, and all smooth spherical symplectic reflection algebras.
(3) All NCCRs that resolve a Gorenstein singularity;
(4) All of the regular algebras discussed by Rogalski are either Calabi-Yau or at least "twisted Calabi-Yau." In particular, the quantum versions of $\mathbf{A}^n$, with
$$x_i x_j = r_{ij} x_j x_i, \quad i < j,$$
are Calabi-Yau if and only if, setting $r_{ji} := r_{ij}^{-1}$ for $i < j$,
$$r_{ij} r_{ji} = 1, \forall i, j, \quad \prod_{j \neq m} r_{mj} = 1, \forall m.$$
For instance, in three variables, we have a single parameter $q = r_{12} = r_{23} = r_{31}$ and then $q^{-1} = r_{21} = r_{32} = r_{13}$.

5.3. **Van den Bergh duality and the BV differential.**

Theorem 5.3. [VdB98] Let A be CY of dimension d. Then, there is a canonical isomorphism, for every A-bimodule M,
$$\mathsf{HH}_i(A, M) \xrightarrow{\sim} \mathsf{HH}^{d-i}(A, M), \tag{5.4}$$
obtained by contraction with a generator of $\mathsf{HH}^d(A, A \otimes A) \cong A$.

More generally, if A is twisted CY with $\mathsf{HH}^d(A, A \otimes A) \cong V_A$, for $\sigma : A \to A$ an algebra automorphism, then $HH_i(A, M \otimes_A V_A) \xrightarrow{\sim} \mathsf{HH}^{d-i}(A, M)$.

Remark 5.5. The above theorem is easy to prove, as we will show, but it is an extremely important observation. Actually Van den Bergh proved in [VdB98] a more difficult result: see Remark 3.30.

Proof. This is a direct computation. For the first statement, using that $\mathsf{HH}^\bullet(A, M) = H^i(\mathrm{RHom}_{A^e}(A, M))$ and similarly $\mathsf{HH}_\bullet(A, M) = A \otimes^{\mathrm{L}}_{A^e} M$,
$$\mathrm{RHom}_{A^e}(A, M) \cong \mathrm{RHom}_{A^e}(A, A \otimes A) \otimes^{\mathrm{L}}_{A^e} M \cong A[-d] \otimes^{\mathrm{L}}_{A^e} M.$$
The homology of the RHS identifies with $\mathsf{HH}_{d-\bullet}(A, M)$ (the degrees were inverted here because $\mathsf{HH}_\bullet$ uses homological grading and $\mathsf{HH}^\bullet$ uses cohomological grading). The remaining statements are similar and left as an exercise. □

Next, recall the HKR theorem: $\mathsf{HH}^\bullet(\mathcal{O}_X) \cong \wedge^\bullet_{\mathcal{O}_X} T_X$ when X is smooth. There is a counterpart for Hochschild homology: $\mathsf{HH}_\bullet(\mathcal{O}_X) \cong \Omega^\bullet_X$, the de Rham complex, viewed with zero differential. Moreover, there is a *homological* explanation of the de Rham differential: this turns out to coincide with the *Connes differential*, which is defined on the Hochschild homology of an *arbitrary* associative algebra.

In the presence of the Van den Bergh duality isomorphism (5.4), this differential B yields a differential on the Hochschild cohomology of a CY algebra,

$$\Delta : \mathsf{HH}^\bullet(A) \to \mathsf{HH}^{\bullet-1}(A).$$

As pointed out in [EG10, §2],

Proposition 5.6. The infinitesimal deformations of a CY algebra A *within the class of CY algebras* are parameterized by $\ker(\Delta) : \mathsf{HH}^2(A) \to \mathsf{HH}^1(A)$.

This is not necessarily the right question to ask, however: if one asks for the deformation space of A *together with its CY structure*, one obtains:

Theorem 5.7. (De Völcsey and Van den Bergh) The infinitesimal deformations of pairs (A, η) where A is a CY algebra and η an A-bimodule quasi-isomorphism in the derived category $D(A^e)$,

$$\eta : C^\bullet(A, A \otimes A) \xrightarrow{\sim} A[-d],$$

is given by the negative cyclic homology $\mathrm{HC}^-_{d-2}(A)$. The obstruction to extending to second order is given by the string bracket $[\gamma, \gamma] \in operatornameHC^-_{d-3}(A)$.

You should think of cyclic homology as a noncommutative analogue of de Rham cohomology, i.e., the homology of $(\mathsf{HH}_\bullet(A), B)$ where B is Connes' differential above (although this is not quite correct).

5.4. **Algebras defined by a potential, and Calabi-Yau algebras of dimension 3.** It turns out that most Calabi-Yau algebras are presented by a *potential*. In the case of dimension three, this has the following beautiful form:

Definition 5.8. Let V be a vector space. A *potential* is an element $\Phi \in TV/[TV, TV]$.

For brevity, from now on we write $TV_{\mathrm{cyc}} := TV/[TV, TV]$.

Definition 5.9. Let $\xi \in \mathrm{Der}(TV)$ be a constant-coefficient vector field, e.g., $\xi = \partial_i$. We define an action of ξ on potentials as follows: for cyclic words $[v_1 \cdots v_m]$ with $v_i \in V$,

$$\xi[v_1 \cdots v_m] = \sum_{i=1}^{m} \xi(v_i) v_{i+1} \cdots v_m v_1 \cdots v_{i-1}.$$

Then we extend this linearly to TV_{cyc}.

Definition 5.10. The algebra A_Φ defined by a potential $\Phi \in TV_{\mathrm{cyc}}$ is

$$A_\Phi := A/(\partial_1 \Phi, \ldots, \partial_n \Phi).$$

Definition 5.11. A potential Φ is called a Calabi-Yau (CY) potential (of dimension 3) if A_Φ is a Calabi-Yau algebra of dimension three.

Example 5.12. For $V = \mathbf{k}^3$, $\mathbf{A}^3$ itself is defined by the potential

$$\Phi = [xyz] - [xzy] \in TV_{\mathrm{cyc}}. \tag{5.13}$$

$$\partial_x \omega = yz - zy, \quad \partial_y \omega = zx - xz, \quad \partial_z \omega = xy - yx.$$

Example 5.14. The universal enveloping algebra $U\mathfrak{sl}_2$ of a Lie algebra is defined by the potential

$$[efh] - [ehf] - \frac{1}{2}[h^2] - [f^2] - [e^2].$$

Example 5.15. The Sklyanin algebra with relations

$$xy - tyx + cz^2 = 0, \quad yz - tzy + cx^2 = 0, \quad zx - xz + cy^2 = 0,$$

from Rogalski's lecture is given by the potential

$$[xyz] - t[xzy] + \frac{c}{3}[x^3 + y^3 + z^3].$$

Example 5.16. NCCR examples? See Wemyss's lectures!

5.5. **Deformations of potentials and PBW theorems.** The first part of the following theorem was proved in the filtered case in [BT07] and in the formal case in [EG10]. I have written informal notes proving the converse (the second part).

Theorem 5.17. Let A_Φ be a graded CY algebra defined by a (CY) potential Φ. Then for any filtered or formal deformation $\Phi + \Phi'$ of Φ, the algebra $A_{\Phi+\Phi'}$ is a filtered or formal CY deformation of A_Φ.

Conversely, all filtered or formal deformations of A_Φ are obtained in this way.

Now, let us return to the setting of $\mathcal{O}_{\mathbf{A}^3} = A_\omega$, which is CY using the potential (5.13). Let $f \in \mathcal{O}_{\mathbf{A}^3}$ be a hypersurface. Then one obtains as above the Poisson structure π_f on $\mathbf{A}^3$ for which f is Poisson central. As above, all quantizations are given by CY deformations $\Phi + \Phi'$. Moreover, by Corollary 4.52, f deforms to a central element, call it $f_{\Phi'}$ of each such quantization. Therefore, one obtains a quantization of the hypersurface $(\mathcal{O}_{\mathbf{A}^3}/(f), \pi_f)$, namely $A_{\Phi+\Phi'}/f_{\Phi'}$.

Next, what we would like to do is to extend this to *graded* deformations, in the case that Φ is homogeneous, i.e., cubic: this will yield the ordinary three-dimensional Sklyanin algebras. More generally, we will consider the *quasihomogeneous* case, which means that it is homogeneous if we assign each of x, y, and z certain weights, which need not be equal. This will recover the "weighted Sklyanin algebras." Note that producing actual graded deformations is not an immediate consequence of the above theorem, and they are not immediately provided by Kontsevich's theorem either. In fact, their existence is a special case (the nicest nontrivial one!) of the following broad conjecture of Kontsevich:

Conjecture 5.18. [Kon01, Conjecture 1] Suppose that π is a quadratic Poisson bivector on $\mathbf{A}^n$, i.e., $\{-,-\}$ is homogeneous, and $\mathbf{k} = \mathbf{C}$. Then, the star-product $\star_\hbar$ in the Kontsevich deformation quantization, up to a suitable formal gauge equivalence, actually converges for $\hbar$ in some complex neighborhood of zero, producing an actual deformation $(\mathcal{O}_{\mathbf{A}^n}, \star_\hbar)$ parameterized by $|\hbar| < \epsilon$, for some $\epsilon > 0$.

As we will see (and was already known, via more indirect constructions), the conjecture holds for π_f with $f \in \mathcal{O}_{\mathbf{A}^3}$ quasihomogeneous such that $Z(f)$ has an isolated singularity at the origin.

5.6. **Etingof-Ginzburg's quantization of isolated quasihomogeneous surface singularities.** In [EG10], Etingof and Ginzburg constructed a (perhaps universal) family of *graded* quantizations of hypersurface singularities in $\mathbf{A}^3$. These essentially coincide with the Sklyanin algebras associated to types E_6, E_7, and E_8 (the E_6 type is the one of Example 5.15).

Namely, suppose that x, y, and z are assigned positive weights $a, b, c > 0$, not necessarily all one. Let $d := a + b + c$. Suppose $f \in \mathbf{k}[x, y, z]$ is a polynomial which is weight-homogeneous of degree m. Then the Poisson bracket π_f (5.1) has weight $m - d$. If we want to therefore deform in such a way as to get a *graded* quantization, we will want to have $m = d$, which is called the *elliptic* case. For a filtered by not necessarily graded quantization, we need only that f be a sum of weight-homogeneous monomials of degree $\leq m$. The case where f is weight-homogeneous of degree strictly less than m turns out to yield all the *du Val* (or Klenian) singularities $\mathbf{A}^2/\Gamma$ for $\Gamma < \mathsf{SL}(2)$ finite, which we discussed before.

So, for a graded quantization, suppose f is homogeneous of degree d. We also need to require that $Z(f)$ has an isolated singularity at the origin. In this case, it is well-known that the possibilities of f, up to isomorphism, are given by a single parameter $\tau \in \mathbf{k}$ together with three possibilities:

$$\frac{1}{3}(x^3+y^3+z^3)+\tau\cdot xyz, \quad \frac{1}{4}x^4+\frac{1}{4}y^4+\frac{1}{2}z^2+\tau\cdot xyz, \quad \frac{1}{6}x^6+\frac{1}{3}y^3+\frac{1}{2}z^2+\tau\cdot xyz.$$

Let p, q, and r denote the exponents: so in the first case, $p=q=r=3$, in the second case $p=q=4$ and $r=2$, and in the third case, $p=6, q=3$, and $r=2$. Note that $ap=bq=cr$; so in the first case we could take $a=b=c=1$; in the second, $a=b=1, c=2$; in the third, $a=1, b=2$, and $c=3$. Let $TV_{\mathrm{cyc}}^{\leq m}$ and TV_{cyc}^{m} be the subspaces of TV_{cyc} of weighted degrees $\leq m$ and exactly m, respectively. Similarly define $TV_{\mathrm{cyc}}^{<m}=TV_{\mathrm{cyc}}^{\leq m-1}$. For a filtered algebra A, we similarly define $A^m, A^{\leq m}$, and $A^{<m}$.

We define μ to be the Milnor number of the singularity of the homogeneous f above at the origin, i.e.,

$$\mu=\frac{(a+b)(a+c)(b+c)}{abc}=p+q+r-1.$$

Finally, we will consider more generally inhomogeneous polynomials replacing f above, of the form $P(x)+Q(y)+R(z)$, such that $\mathsf{gr}(P(x)+Q(y)+R(z))=f$ and $P(0)=Q(0)=R(0)=0$. Clearly such f have μ parameters.

Similarly, we will consider potentials $\Phi_{P,Q,R}^{t,c}$ of the form

$$\Phi_{P,Q,R}^{t,c}:=[xyz]-t[yxz]+c(P(x)+Q(y)+R(z)). \tag{5.19}$$

If we set $t:=e^{\tau\hbar}$, then working over $\mathbf{k}[\![\hbar]\!]$, with $c\in\mathbf{k}$ and $P(x), Q(y), R(z)$ polynomials as above, the algebra $A_{\Phi_{P,Q,R}^{t,c}}$ is a deformation quantization of $\mathcal{O}_{\mathbf{A}^3}$, equipped with the Poisson bracket

$$\pi_f, \quad f=\tau\cdot xyz-c\big(P(x)+Q(y)+R(z)\big).$$

Thus, for $\mathbf{k}=\mathbf{C}$, the graded algebras $A_{\Phi_{P,Q,R}^{t,c}}$ indeed give graded quantizations (for actual values of $\hbar$) of the Poisson structures π_f associated to all quasihomogeneous (when P, Q, R are quasihomogeneous), and more generally all filtered polynomials f of weighted degree $\leq d$. This is actually true for arbitrary values of $a, b, c \geq 1$ with $d=a+b+c$, without requiring that the homogeneous part of f have an isolated singularity at zero. This confirms Kontsevich's conjecture 5.18 in these cases, with $\hbar\in\mathbf{C}$ convergent everywhere, up to the fact that $A_{\Phi_{P,Q,R}^{e^{\hbar},c}}$ need not (à priori) correspond precisely to the Poisson structure $\hbar\cdot\pi_f$, but rather to some formal Poisson deformation thereof.

However, in this case, since the f above are all possible filtered polynomials of degree $\leq d$, we can conclude that, for every $f_\hbar\in\hbar\cdot\mathcal{O}_{\mathbf{A}^3}^{\leq d}[\![\hbar]\!]$, letting $\star_{f_\hbar}$ be the Kontsevich quantization of $\pi_{f_\hbar}$, there must exist $\Phi_\hbar\in TV_{\mathrm{cyc}}^{\leq d}[\![\hbar]\!]$ such that

$$(\mathcal{O}_{\mathbf{A}^3}[\![\hbar]\!], \star_{f_\hbar})\cong A_{\Phi_\hbar}. \tag{5.20}$$

Probably, one can show that the resulting isomorphism preserves the subset of formal deformations that (up to formal gauge equivalence) converge in some neighborhood of $\hbar=0$, and this would be enough to confirm Conjecture 5.18.

Question 5.21. What is the relation between the parameters $f_\hbar$ and $\Phi_\hbar$ above? In particular:

(1) Does (5.20) send star products that converge for $\hbar$ in some neighborhood of zero to potentials that also converge in some neighborhood of zero?
(2) Is $A_{\Phi_{P,Q,R}^{e^{\tau\cdot\hbar},c}}$ isomorphic to Kontsevich's quantization of $\hbar\cdot\pi_f$ (or some constant multiple), for $f=\tau\cdot xyz-c\big(P(x)+Q(y)+R(z)\big)$?

Returning to the case that the degrees (a, b, c) are in the set $\{(1,1,1),(1,1,2),(1,2,3)\}$, i.e., that generic quasihomogeneous polynomials of degree d have an isolated singularity at the origin, the main result of [EG10] shows that, for generic A_Φ with Φ filtered of degree $\leq d = a+b+c$, the algebra A_Φ is in the family $A_{\Phi^{t,c}_{P,Q,R}}$ and the family provides a versal deformation; moreover, the center is a polynomial algebra in a single generator, and quotienting by this generator produces a quantization of the original isolated singularity, which also restricts to a versal deformation generically:

Theorem 5.22. [EG10]

(i) Suppose that $\Phi \in TV^d_{\text{cyc}}$ is a homogeneous CY potential of weighted degree $d = a + b + c$, where $|x| = a$, $|y| = b$, and $|z| = c$ (which can be arbitrary). Then for any potential $\Phi' \in TV^{<d}_{\text{cyc}}$ of degree strictly less than d, $\Phi + \Phi'$ is also a CY potential. Moreover, the Hilbert series of the CY algebra $A_{\Phi+\Phi'}$ is

$$h(A_{\Phi+\Phi'};t) = \frac{1}{(1-t^a)(1-t^b)(1-t^c)}.$$

(ii) There exists a nonscalar central element $\Psi \in A^{\leq d}_{\Phi+\Phi'}$.

(iii) Now suppose that $(a, b, c) \in \{(1,1,1),(1,1,2),(1,2,3)\}$ as above. Then, if Φ is generic, then there exist parameters $P, Q, R, t,\ c$, as above, such that for all $\Phi' \in TV^{<d}_{\text{cyc}}$,

$$A_{\Phi+\Phi'} \cong A_{\Phi^{t,c}_{P,Q,R}}.$$

Moreover, in this case, $Z(A_{\Phi+\Phi'}) = \mathbf{k}[\Psi]$ is a polynomial algebra in one variable, and $Z(\mathsf{gr}\, A_{\Phi+\Phi'}) = \mathbf{k}[\mathsf{gr}\,\Psi]$.

(iv) Keep the assumption of (iii). The family $\{A_{\Phi^{t',c'}_{P',Q',R'}}\}$ restricts to a versal deformation of $A_{\Phi+\Phi'}$ in the formal neighborhood of (P, Q, R, t, c), depending on μ parameters. Moreover, the family $\{A_{\Phi^{t',c'}_{P',Q',R'}}/\Psi^{t',c'}_{P',Q',R'}\}$ restricts to a versal deformation of $A_{\Phi+\Phi'}/(\Psi)$ in the same formal neighborhood.

Question 5.23. Note that we did *not* say anything above about the family $A_{\Phi^{t,c}_{P,Q,R}}/(\Psi^{t,c}_{P,Q,R})$ restricting to a versal quantization of the original singularity $\mathcal{O}_{\mathbf{A}^3}/(f)$ for $f = \tau \cdot xyz - c\big(P(x) + Q(y) + R(z)\big)$. More precisely, does the family of central reductions,

$$A^{t_\hbar,\hbar\cdot c_\hbar}_{\Phi_{\hbar\cdot P_\hbar,\hbar\cdot Q_\hbar,\hbar\cdot R_\hbar}}/(\Psi^{t_\hbar,\hbar\cdot c_\hbar}_{\hbar\cdot P_\hbar,\hbar\cdot Q_\hbar,\hbar\cdot R_\hbar}),$$

produce a versal deformation quantization, for $P_\hbar(x), Q_\hbar(y), R_\hbar(z) \in \mathcal{O}_{\mathbf{A}^3}[\![\hbar]\!]$ and $t_\hbar, c_\hbar \in \mathbf{k}[\![\hbar]\!]$ such that $P = P_0, Q = Q_0, R = R_0, c = c_0$, and $t_\hbar \equiv e^{\tau\hbar} \pmod{\hbar^2}$?

In the case that P, Q, and R are homogeneous, i.e., $f = P(x) + Q(y) + R(z)$, we can explicitly write out the central elements Ψ for the first two possibilities for f ([EG10, (3.5.1)]). For the first one, $f = \frac{1}{3}(x^3 + y^3 + z^3) + \tau \cdot xyz$, we get

$$\Psi = c \cdot y^3 + \frac{t^3 - c^3}{c^3 + 1}(yzx + c \cdot z^3) - t \cdot zyx. \tag{5.24}$$

For the second one, $\frac{1}{4}x^4 + \frac{1}{4}y^4 + \frac{1}{2}z^2 + \tau \cdot xyz$, we get

$$\Psi = (t^2 + 1)xyxy - \frac{t^4 + t^2 + 1}{t^2 - c^4}(t \cdot xy^2x + c^2 \cdot y^4) + t \cdot y^2x^2. \tag{5.25}$$

For the third case, the answer is too long, but you can look it up at `http://www-math.mit.edu/~etingof/delpezzocenter`. Moreover, there one can obtain the formulas for the central elements Ψ in the filtered cases as well.

Appendix A. Exercise Sheet One

(1) The exercises from the notes:
   - Exercise 1.9.
   - Exercise 1.13.
   - Exercise 1.14.
   - Exercise 1.23.
   - Exercise 1.25.
   - Exercise 1.32.

(2) (a) Verify, using Singular, that $\mathsf{Weyl}(\mathbf{k}^2)^{\mathbf{Z}/2} \cong (U\mathfrak{sl}_2)_\eta$ where $\eta(C) = \eta(ef+fe+\frac{1}{2}h^2) = \frac{-3}{8}$.
How to do this: Type the commands

```
LIB "nctools.lib";
def a = makeWeyl(1);
setring a;
a;
```

Now you can play with the Weyl algebra with variables D and x; D corresponds to ∂_x.
Now you need to figure out some polynomials $e(x,D)$, $f(x,D)$, and $h(x,D)$ so that

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$$

Then, once you have done this, compute the value of the Casimir,

$$C = ef + fe + \frac{1}{2}h^2,$$

and verify it is $\frac{-3}{8}$.
Hint: the polynomials $e(x,D)$, $f(x,D)$, and $h(x,D)$ should be homogeneous quadratic polynomials (that way the bracket is linear).
For example, try first

```
def e=D^2;
def f=x^2;
def h=e*f-f*e;
h*e-e*h;
2*e;
h*f-f*h;
2*f;
```

and you see that, defining e and f as above and h to be $ef - fe$ as needed, we don't quite get $he - eh = 2e$ or $hf - fh = 2f$. But you can correct this...

(b) Show that the highest weight of a highest weight representation of $(U\mathfrak{sl}_2)_\eta$, for this η, is either $-\frac{3}{2}$ or $-\frac{1}{2}$. Here, a highest weight representation is one generated by a vector v such that $e \cdot v = 0$ and $h \cdot v = \mu v$ for some $\mu \in \mathbf{k}$. Then, μ is called its highest weight. The value $-\frac{1}{2}$ is halfway between the value 0 of the highest weight of the trivial representation of $\mathfrak{sl}_2$ and the value -1 of the highest weight of the Verma module for the unique χ such that $(U\mathfrak{sl}_2)_\chi$ has infinite Hochschild dimension.

Remark A.1. Here, a Verma module of $\mathfrak{sl}_2$ is a module of the form $U\mathfrak{sl}_2/(e, h-\mu)$ for some μ; this Verma module of highest weight -1 is also the unique one that has different central character from all other Verma modules, since the action of the Casimir on a Verma with highest weight μ is by $\frac{1}{2}\mu^2 + \mu$, so C acts by multiplication by $\frac{-1}{2}$ on this Verma and no others. Therefore, the category $\mathcal{O}$ defined in Bellamy's lectures for the central quotient $U\mathfrak{sl}_2/(C + \frac{1}{2})$ is equivalent to the category of vector spaces, with this Verma as the unique simple object. This is the only central quotient with this property. Also, the corresponding Cherednik algebra considered in Bellamy's lectures, of which $U\mathfrak{sl}_2/(C+\frac{1}{2})$ is the spherical subalgebra, has semisimple category $\mathcal{O}$ (now with two simple Verma modules: one of them which is killed by symmetrization, so only one yields a module over the spherical subalgebra).

(c) Identify the representation $\mathbf{k}[x^2]$ over $\mathsf{Weyl}(\mathbf{k}^2)^{\mathbf{Z}/2}$ with a highest weight representation of $(U\mathfrak{sl}_2)_\eta$.

(d) Use the isomorphism $(U\mathfrak{sl}_2)_\eta$ and representation theory of $\mathfrak{sl}_2$ to show that $\mathsf{Weyl}(V)^G$ admits no finite-dimensional irreducible representations (cf. Exercise 4: in fact $\mathsf{Weyl}(V)^G$ is simple). Hence it admits no finite-dimensional representations at all.

(3) The key to deformation theory is the notion of *flatness*, i.e., given an algebra B, we are always interested in an algebra A deforming B such that A is the same size as B: precisely, this is what we want if B is graded and A is supposed to be a filtered deformation (so $B = \mathsf{gr}\, A$ implies they have the same size, i.e., the same dimension or more generally the same Hilbert series when these exist; in general if we deform the relations defining B, we always get a canonical surjection $B \twoheadrightarrow \mathsf{gr}\, A$ and we want this to be injective, i.e., an isomorphism). We will have other notions of flatness when we deform B over a ring (e.g., a flat formal deformation A would be a deformation over $\mathbf{k}[\![\hbar]\!]$ which is isomorphic to $B[\![\hbar]\!]$ as a $\mathbf{k}[\![\hbar]\!]$-module; if it is defined by relations and is not flat, that would mean that it instead has $\hbar$ torsion.)

Now we use GAP to try to play with flat deformations. Here is code to get you started for the universal enveloping algebra of $\mathfrak{sl}_2$:

```
LoadPackage("GBNP");
A:=FreeAssociativeAlgebraWithOne(Rationals, "e", "f", "h");
e:=A.e;; f:=A.f;; h:=A.h;; o:=One(A);;
uerels:=[f*e-e*f+h,h*e-e*h-2*e,h*f-f*h+2*f];
uerelsNP:=GP2NPList(uerels);;
PrintNPList(uerelsNP);
GBNP.ConfigPrint(A);
GB:=SGrobner(uerelsNP);;
PrintNPList(GB);
```

This computes the Gröbner basis for the ideal generated by the relations. You can also get explicit information about how each Gröbner basis element is obtained from the original relations:

```
GB:=SGrobnerTrace(uerelsNP);;
PrintNPListTrace(GB);
PrintTracePol(GB[1]);
```

Here the second line gives you the list of Gröbner basis elements, and for each element $GB[i]$, the third line will tell you how it is expressed in terms of the original relations.

(a) Verify that, with the $U\mathfrak{sl}_2$ relations, $[x,y]=z, [y,z]=x, [z,x]=y$, one gets a flat filtered deformation of $\mathsf{Sym}\langle x,y,z\rangle$, by looking at the Gröbner basis.

(b) Now play with modifying those relations and see that, for most choices of filtered deformations, one need not get a flat deformation.

(c) Now play with the simplest Cherednik algebra: the deformation of $\mathsf{Weyl}(\mathbf{k}^2) \rtimes \mathbf{Z}/2$. The algebra $\mathsf{Weyl}(\mathbf{k}^2) \rtimes \mathbf{Z}/2$ itself is defined by relations $x*y-y*x-1$ (the Weyl algebra relation), together with z^2-1 (so z generates $\mathbf{Z}/2$) and $x*z+z*x, y*z+z*y$, so z anticommutes with x and y. S
Then, the deformation is given by replacing the relation $x*y-y*x-1$ with the relation $x*y-y*x-1-\lambda\cdot z$, for $\lambda \in \mathbf{k}$ a parameter. Show that this is a flat deformation of $\mathsf{Weyl}(\mathbf{k}^2) \rtimes \mathbf{Z}/2$ (again you can just compute the Gröbner basis).

(d) Now modify the action of $\mathbf{Z}/2$ so as to not preserve the symplectic form: change $y * z + z * y$ into $y * z - z * y$. What happens to the algebra defined by these relations now?

(4) Verify that, for the Weyl algebra, over a field $\mathbf{k}$ of characteristic zero, we have a Morita equivalence $\mathsf{Weyl}(V) \rtimes G \simeq \mathsf{Weyl}(V)^G$. Show that $\mathsf{Weyl}(V) \rtimes G$ is simple, and use the Morita equivalence to conclude that $\mathsf{Weyl}(V)^G$ is simple. Use that $\mathsf{Weyl}(V)$ is a simple algebra, i.e., it has no nonzero proper two-sided ideals. More generally, show that if A is a simple algebra over a field $\mathbf{k}$ of characteristic zero and G is a finite group acting by automorphisms, then A^G is Morita equivalent to $A \rtimes G$.

Hint: in general, show that, if e is an idempotent of a ring B, then eBe is Morita equivalent to B if $B = BeB$, using the (eBe, B) bimodule eB and the (B, eBe) bimodule Be, since the condition guarantees that these are projective as left and right modules and that $Be \otimes_{eBe} eB = B$ and $eB \otimes_B Be = eBe$.

Then let e be the symmetrizer of $\mathbf{k}[G]$, and show that $AeA = A$ if A is simple.

(5) Given an algebra B, to understand its Hochschild cohomology (which governs deformations of associative algebras!) as well as many things, we need to construct resolutions. A main tool then is *deformations* of resolutions.

Suppose that B is weight-graded and that A is a filtered deformation (i.e., $\mathsf{gr}\, A = B$). Let M be a filtered A-module. Suppose that $Q_\bullet$ is a filtered complex of projectives such that $\mathsf{gr}\, Q_\bullet$ is a projective resolution of $\mathsf{gr}\, M$. Show that $Q_\bullet$ is a projective resolution of $H_0(Q_\bullet)$.

Moreover, if the direct sum of all Q_i, $\bigoplus_i Q_i$, is finite-dimensional in each weight-graded component, show that $H_0(Q_\bullet) = M$, i.e., $Q_\bullet$ is a projective resolution of M.

Hint: Show that $\dim H_i(\mathsf{gr}\, C_\bullet) \geq \dim H_i(C_\bullet)$ for an arbitrary filtered complex $C_\bullet$, because $\ker(\mathsf{gr}\, C_i \to \mathsf{gr}\, C_{i+1}) \supseteq \mathsf{gr}(\ker C_i \to C_{i+1})$ and $\operatorname{im}(\mathsf{gr}\, C_{i-1} \to \mathsf{gr}\, C_i) \subseteq \mathsf{gr}\, \operatorname{im}(C_{i-1} \to C_i)$.

For the final statement, apply the fact that the weight-graded Hilbert series of $H_\bullet(Q_\bullet)$, i.e., the alternating sum $\sum_i (-1)^i h(H_i(Q_\bullet); t)$, taking Hilbert series with respect to the weight grading, equals $h(Q_\bullet; t)$ (by definition this is $\sum_i (-1)^i h(Q_i; t)$), and hence also equals $h(\mathsf{gr}\, Q_\bullet; t)$, and hence also equals $h(\mathsf{gr}\, H_\bullet(Q_\bullet); t)$.

Corollary A.2. Suppose that $B = TV/(R)$ for some homogeneous relations $R \subseteq R$, and that B has a finite free B-bimodule resolution, $P_\bullet \twoheadrightarrow B$. If $E \subseteq TV$ is a deformation of the homogeneous relations R, i.e., $\mathsf{gr}\, E = R$, then the following are equivalent:

(i) $A := TV/(E)$ is a flat filtered deformation of B, i.e., the canonical surjection $B \twoheadrightarrow \mathsf{gr}\, A$ is an isomorphism;

(ii) The resolution $P_\bullet$ deforms to a filtered free resolution $Q_\bullet$ of A.

In this case, the deformed complex in (ii) is a free resolution of A.

Proof. To show (ii) implies (i), take the graded Euler characteristic of the complex; this implies that $\mathsf{gr}\, A$ and B have the same graded Euler characteristic, so $B \twoheadrightarrow \mathsf{gr}\, A$ is an isomorphism.

To show (i) implies (ii), inductively construct a deformation of $P_\bullet$ to a filtered free resolution of A. First of all, we know that $P_0 \twoheadrightarrow B$ deforms to $Q_0 \twoheadrightarrow A$, since we can arbitrarily lift the map $P_0 = B^{r_0} \to B$ to a filtered map $Q_0 := A^{r_0} \to A$, and this must be surjective since $B \twoheadrightarrow \mathsf{gr}\, A$ is surjective. Since in fact it is an isomorphism, the Hilbert series of the kernels are the same, and so we can arbitrarily lift the surjection $P_1 = B^{r_1} \twoheadrightarrow \ker(P_0 \to B)$ to a filtered map $Q_1 := A^{r_1} \twoheadrightarrow \ker(Q_0 \to A)$, etc. Fill in the details! ∎

In the case B is Koszul, we can do better using the form of the Koszul complex:

Corollary A.3. (Koszul deformation principle) If B is a Koszul algebra presented as $TV/(R)$ for $R \subseteq V^{\otimes 2}$, and $E \subseteq V^{\otimes \leq 2}$ is a filtered deformation of R (i.e., $\mathsf{gr}\, E = R$), then $A := TV/(E)$ is a flat deformation of B if and only if it is flat in filtered degree three, i.e., if and only if the canonical surjection $B_{\leq 3} \twoheadrightarrow \mathsf{gr}\, A_{\leq 3}$ is an isomorphism.

Similarly, if $E \subseteq V^{\otimes 2}[\![\hbar]\!]$ is a formal deformation with quadratic relations, then $A := TV[\![\hbar]\!]/(E)$ is a flat formal deformation of B if and only if it is flat in graded degree three, i.e., $A^3 \cong B^3[\![\hbar]\!]$ as a $\mathbf{k}[\![\hbar]\!]$-module.

Proof. For you to do if you know Koszul complexes (or look at, e.g., [BGS96]: the key is that flatness in degree ≤ 3 (or equal to 3 in the second case) is equivalent to saying that the Koszul complex of B deforms to a complex; then we apply the exercise. □

(6) Next, we play with Koszul complexes, which are one of the main tools for computing Hochschild cohomology.

First consider, for the algebra $\mathsf{Sym}\, V$, the Koszul resolution of the augmentation module $\mathbf{k}$:

$$0 \to \mathsf{Sym}\, V \otimes \wedge^{\dim V} V \to \mathsf{Sym}\, V \otimes \wedge^{\dim V - 1} V \to \cdots \to \mathsf{Sym}\, V \otimes V \to \mathsf{Sym}\, V \twoheadrightarrow \mathbf{k}, \tag{A.4}$$

$$f \otimes (v_1 \wedge \cdots \wedge v_i) \mapsto \sum_{j=1}^{i} (-1)^{j-1} (f v_j) \otimes (v_1 \wedge \cdots \hat{v}_j \cdots \wedge v_i), \tag{A.5}$$

where $\hat{v}_j$ means that v_j was omitted from the wedge product.

(a) Construct from this a bimodule resolution of $\mathsf{Sym}\, V$,

$$\mathsf{Sym}\, V \otimes \wedge^{\bullet} V \otimes \mathsf{Sym}\, V \twoheadrightarrow \mathsf{Sym}\, V.$$

Using this complex, show that $\mathsf{HH}^i(\mathsf{Sym}\, V) := \mathrm{Ext}^i_{(\mathsf{Sym}\, V)^e}(\mathsf{Sym}\, V, \mathsf{Sym}\, V) \cong \mathsf{Sym}\, V[-\dim V]$. That is, $\mathsf{Sym}\, V$ is a Calabi-Yau algebra of dimension $\dim V$ (recall the definition, mentioned in Rogalski's lecture, from Definition 2.37 of the notes).

(b) Now replace $\mathsf{Sym}\, V$ by the Weyl algebra, $\mathsf{Weyl}(V)$. Show that the Koszul complex above deforms to give a complex whose zeroth homology is $\mathsf{Weyl}(V)$:

$$\begin{aligned} 0 \to \mathsf{Weyl} V \otimes \wedge^{\dim V} V \otimes \mathsf{Weyl} V \to \mathsf{Weyl} V \otimes \wedge^{\dim V - 1} V \otimes \mathsf{Weyl} V \\ \to \cdots \to \mathsf{Weyl} V \otimes V \otimes \mathsf{Weyl} V \to \mathsf{Weyl} V \otimes \mathsf{Weyl} V \twoheadrightarrow \mathsf{Weyl} V, \end{aligned} \tag{A.6}$$

using the same formula. (You only need to show it is a complex, by the previous exercise. Note also the fact that such a deformation exists is a consequence of the corollaries of the previous exercise.)

Deduce that $\mathsf{Weyl} V$ is also CY of dimension $\dim \mathsf{Weyl} V$.

(c) Suppose that $V = \mathfrak{g}$ is a (finite-dimensional) Lie algebra. Deform the complex to a resolution of the universal enveloping algebra using the Chevalley-Eilenberg complex:

$$U\mathfrak{g} \otimes \wedge^{\dim \mathfrak{g}} \mathfrak{g} \otimes U\mathfrak{g} \to U\mathfrak{g} \otimes \wedge^{\dim \mathfrak{g} - 1} \mathfrak{g} \otimes U\mathfrak{g} \to \cdots \to U\mathfrak{g} \otimes \mathfrak{g} \otimes U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g} \twoheadrightarrow U\mathfrak{g},$$

where the differential is the sum of the preceding differential for $\mathsf{Sym}\, V = \mathsf{Sym}\, \mathfrak{g}$ and the additional term,

$$x_1 \wedge \cdots \wedge x_k \mapsto \sum_{i<j} [x_i, x_j] \wedge x_1 \wedge \cdots \hat{x}_i \cdots \hat{x}_j \cdots \wedge x_k.$$

That is, verify this is a complex, and conclude from the (a) and the preceding exercise that it must be a resolution.

Conclude that $U\mathfrak{g}$ is also Calabi-Yau of dimension $\dim \mathfrak{g}$.

Moreover, recall the usual Chevalley-Eilenberg complexes computing Lie algebra (co)homology with coefficients in a $\mathfrak{g}$-module M: if $C_\bullet^{CE}(\mathfrak{g})$ is the complex inside of the two copies of $U\mathfrak{g}$ above, these are

$$C_\bullet^{CE}(\mathfrak{g}, M) := C_\bullet^{CE}(\mathfrak{g}) \otimes M, \quad C^\bullet_{CE}(\mathfrak{g}, M) := \mathrm{Hom}_{\mathbf{k}}(C_\bullet^{CE}(\mathfrak{g}), M).$$

Conclude that (cf. [Lod98, Theorem 3.3.2]), if M is a $U\mathfrak{g}$-bimodule,

$$\mathsf{HH}^\bullet(U\mathfrak{g}, M) \cong H^\bullet_{CE}(\mathfrak{g}, M^{\mathrm{ad}}), \quad \mathsf{HH}_\bullet(U\mathfrak{g}, M) \cong H_\bullet^{CE}(\mathfrak{g}, M^{\mathrm{ad}}),$$

where M^{ad} is the $\mathfrak{g}$-module obtained from M by the adjoint action,

$$\mathrm{ad}(x)(m) := xm - mx,$$

where the LHS gives the Lie action of x on m, and the RHS uses the $U\mathfrak{g}$-bimodule action.

(d) **Corrected !!** Conclude from the final assertion of (c) that $\mathsf{HH}^\bullet(U\mathfrak{g}) = H^\bullet_{CE}(\mathfrak{g}, U\mathfrak{g}^{\mathrm{ad}})$. Next, recall (or accept as a black box) that, when $\mathfrak{g}$ is finite-dimensional semisimple (or more generally reductive), then $U\mathfrak{g}^{\mathrm{ad}}$ decomposes into a direct sum of finite-dimensional irreducible representations, and $H^\bullet_{CE}(\mathfrak{g}, V) = \mathrm{Ext}^\bullet_{U\mathfrak{g}}(\mathbf{k}, V) = 0$ if V is finite-dimensional irreducible and *not* the trivial representation.
Conclude that $\mathsf{HH}^\bullet(U\mathfrak{g}) = \mathrm{Ext}^\bullet_{U\mathfrak{g}}(\mathbf{k}, U\mathfrak{g}^{\mathrm{ad}}) = Z(U\mathfrak{g}) \otimes \mathrm{Ext}^\bullet_{U\mathfrak{g}}(\mathbf{k}, \mathbf{k})$.
Bonus: Compute that $\mathrm{Ext}_{U\mathfrak{g}}(\mathbf{k}, \mathbf{k}) = H^\bullet_{CE}(\mathfrak{g}, \mathbf{k})$ equals $(\wedge^\bullet \mathfrak{g}^*)^{\mathfrak{g}}$. To do so, show that the inclusion $C^\bullet_{CE}(\mathfrak{g}, \mathbf{k})^{\mathfrak{g}} \hookrightarrow C^\bullet_{CE}(\mathfrak{g}, \mathbf{k})$ is a quasi-isomorphism. This follows by defining the oeprator d^* with opposite degree of d, so that the Laplacian $dd^* + d^*d$ is the quadratic Casimir element $C = \sum_i e_i f_i + f_i e_i + \frac{1}{2}h_i^2$. Thus this Laplacian operator defines a contracting homotopy onto the part of the complex on which the quadratic Casimir acts by zero. Since C acts by a positive scalar on all nontrivial finite-dimensional irreducible representations, this subcomplex is $C^\bullet_{CE}(\mathfrak{g}, \mathbf{k})^{\mathfrak{g}}$. But the latter is just $(\wedge \mathfrak{g}^*)^{\mathfrak{g}}$ with zero differential.
Conclude that $\mathsf{HH}^\bullet(U\mathfrak{g}) = Z(U\mathfrak{g}) \otimes (\wedge^\bullet \mathfrak{g}^*)^{\mathfrak{g}}$. In particular, $\mathsf{HH}^2(U\mathfrak{g}) = \mathsf{HH}^1(U\mathfrak{g}) = 0$, so $U\mathfrak{g}$ has no nontrivial formal deformations.

Remark A.7. or $\mathfrak{g} = \mathfrak{sl}_2$, $U\mathfrak{sl}_2$ is a Calabi-Yau algebra, since it is a Calabi-Yau deformation of the Calabi-Yau algebra $\mathcal{O}_{\mathfrak{sl}_2^*} \cong \mathbf{A}^3$, which is Calabi-Yau with the usual volume form. So $\mathsf{HH}^3(U\mathfrak{sl}_2) \cong \mathsf{HH}_0(U\mathfrak{sl}_2)$ by Van den Bergh duality, and the latter is $U\mathfrak{sl}_2/[U\mathfrak{sl}_2, U\mathfrak{sl}_2]$, which you should be able to prove equals $(U\mathfrak{sl}_2)_{\mathfrak{sl}_2} \cong (U\mathfrak{sl}_2)^{\mathfrak{sl}_2}$, i.e., this is a rank-one free module over $\mathsf{HH}^0(U\mathfrak{sl}_2)$.
In other words, $H^3_{CE}(\mathfrak{g}, \mathbf{k}) \cong \mathbf{k}$, which is also true for a general semisimple Lie algebra. Since this group controls the *tensor category deformations* of $\mathfrak{g}$-mod, or the Hopf algebra deformations of $U\mathfrak{g}$, this is saying that there is a one-parameter deformation of $U\mathfrak{g}$ as a Hopf algebram which gives the quantum group $U_q\mathfrak{g}$. But since $\mathsf{HH}^2(U\mathfrak{g}) = 0$, as an associative algebra, this deformation is trivial, i.e., equivalent to the original algebra; this is *not* true as a Hopf algebra !

(7) Using the Koszul deformation principle (Corollary A.3), show that the symplectic reflection algebra (Example 2.54 of the notes) is a flat deformation of $\mathsf{Weyl}(V) \rtimes G$. Hint: what Corollary A.3 shows is that you only need to verify that the deformation is flat in degrees ≤ 3, i.e., that the "Jacobi" identity is satisfied,

$$[\omega_s(x, y), z] + [\omega_s(y, z), x] + [\omega_s(z, x), y] = 0,$$

for all symplectic reflections $s \in S$, and all $x, y, z \in V$, as well as the fact that the relations are G-invariant. Cf. [Eti05, §2.3], where it is also shown that the SRA relations are the unique choice of filtered relations which yield a flat deformation.

Appendix B. Exercise Sheet Two

(1) Exercise 2.8.
(2) Exercise 3.31.
(3) Exercise 3.33.

Appendix C. Exercise Sheet Three

(1) Exercise 4.30.
(2) In this exercise we compute the Hochschild (co)homology of a skew group ring.

Let A be an associative algebra over a field $\mathbf{k}$ of characteristic zero, and Γ a finite group acting on A by automorphisms. Form the algebra $A \rtimes \Gamma$, which as a vector space is $A \otimes \mathbf{k}[\Gamma]$, with the multiplication

$$(a_1 \otimes g_1)(a_2 \otimes g_2) = (a_1 g_1(a_2) \otimes g_1 g_2).$$

Next, given any Γ-module N, let $N^\Gamma := \{n \in N \mid g \cdot n = n \forall g \in \Gamma\}$, and $N_\Gamma := N/\{n - g \cdot n \mid n \in N, g \in \Gamma\}$ be the invariants and coinvariants, respectively.

(a) Let M be an $A \rtimes \Gamma$-module. Prove that

$$\mathsf{HH}^\bullet(A \rtimes \Gamma, M) \cong \mathsf{HH}^\bullet(A, M)^\Gamma, \quad \mathsf{HH}_\bullet(A \rtimes \Gamma, M) \cong \mathsf{HH}_\bullet(A, M)_\Gamma,$$

where in the RHS, Γ acts on A and M via the adjoint action, $g \cdot_{\mathrm{Ad}} m = (gmg^{-1})$.
Hint: Write the first one as $\mathrm{Ext}^\bullet_{A^e \rtimes (\Gamma \times \Gamma)}(A \rtimes \Gamma, M)$, using that $\mathbf{k}[\Gamma] \cong \mathbf{k}[\Gamma^{\mathrm{op}}]$ via the map $g \mapsto g^{-1}$. Notice that $A \rtimes \Gamma = \mathrm{Ind}_{A^e \rtimes \Gamma_\Delta}^{A^e \rtimes (\Gamma \times \Gamma)} A$, where $\Gamma_\Delta := \{(g, g) \mid g \in \Gamma\} \subseteq \Gamma \times \Gamma$ is the diagonal subgroup. Then, there is a general fact called Shapiro's lemma, for $H < K$ a subgroup,

$$\mathrm{Ext}_{\mathbf{k}[K]}(\mathrm{Ind}_H^K M, N) \cong \mathrm{Ext}_{\mathbf{k}}(M, N).$$

Using this, show that

$$\mathrm{Ext}^\bullet_{A^e \rtimes (\Gamma \times \Gamma)}(A \rtimes \Gamma, M) \cong \mathrm{Ext}^\bullet_{A^e \rtimes \Gamma_\Delta}(A, M) = \mathrm{Ext}^\bullet_{A^e \rtimes \Gamma}(A^{\mathrm{Ad}}, M^{\mathrm{Ad}}),$$

where A^{Ad} and M^{Ad} mean that Γ acts by the adjoint action from the $A \rtimes \Gamma$-bimodule structure, and on A this is just the original Γ-action. Since taking Γ-invariants is an exact functor (as $\mathbf{k}$ has characteristic zero and Γ is finite), this says that the RHS above equals

$$\mathrm{Ext}^\bullet_{A^e}(A^{\mathrm{Ad}}, M^{\mathrm{Ad}})^\Gamma = \mathsf{HH}^\bullet(A, M)^\Gamma.$$

The proof for Hochschild homology is essentially the same, using Tor.

(b) Now we apply the formula in part (a) to the special case $M = A \rtimes \Gamma$ itself.
Let C be a set of representatives of the conjugacy classes of Γ: that is, $C \subseteq \Gamma$ and for every element $g \in \Gamma$, there exists a unique $h \in C$ such that g is conjugate to h. For $g \in \Gamma$, let $Z_g(\Gamma) < \Gamma$ be the centralizer of g, i.e., the collection of elements that commute with g. Prove that

$$\mathsf{HH}^\bullet(A \rtimes \Gamma) \cong \bigoplus_{h \in C} \mathsf{HH}^\bullet(A, A \cdot h)^{Z_h(\Gamma)},$$

Here, the bimodule action of A on $A \cdot h$ is by

$$a(b \cdot h) = ab \cdot h, \quad (b \cdot h)a = (bh(a)) \cdot h,$$

and $Z_h(\Gamma)$ acts by the adjoint action.

(c) Now specialize to the case that $A = \mathsf{Sym}\, V$ and $\Gamma < \mathsf{GL}(V)$. We will prove here that

$$\mathsf{HH}^\bullet(A \rtimes G) \cong \bigoplus_{h \in C} \left(\left(\wedge_{\mathsf{Sym}\, V^h} T_{(V^h)^*} \right) \otimes \left(\wedge^{\dim (V^h)^\perp} \langle \partial_\phi \rangle_{\phi \in (V^h)^\perp} \right) \right)^{Z_h(\Gamma)}.$$

The degree $\bullet$ on the LHS is the total degree of polyvector field on the RHS, i.e., the sum of the degree in the first exterior algebra with $\dim(V^h)^\perp = \dim V - \dim V^h$. Similarly, the same argument shows

$$\mathsf{HH}_\bullet(A \rtimes G) \cong \bigoplus_{h \in C} \left(\mathsf{Sym}\, V^h \otimes \left(\wedge^{\dim (V^h)^\perp} d(((V^h)^\perp)^*) \right) \right)_{Z_h(\Gamma)}.$$

Hints: first, up to conjugation, we can always assume h is diagonal (since Γ is finite). Suppose that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of h on the diagonal. Then let $h_1, \ldots, h_n \in \mathsf{GL}_1$ be the one-by-one matrices $h_i = (\lambda_i)$. Show that

$$A = \mathbf{k}[x_1] \otimes \cdots \otimes \mathbf{k}[x_n], \quad A \cdot h = (\mathbf{k}[x_1] \cdot h_1) \otimes \cdots \otimes (\mathbf{k}[x_n] \otimes h_n).$$

Conclude using the Künneth formula, $\mathsf{HH}^\bullet(A \otimes B, M \otimes N) = \mathsf{HH}^\bullet(A, M) \otimes \mathsf{HH}^\bullet(B, N)$ (for M an A-bimodule and N a B-bimodule), that

$$\mathsf{HH}^\bullet(A, A \cdot h)^{Z_h(\Gamma)} \cong \bigotimes_{i=1}^{n} \mathsf{HH}^\bullet(\mathbf{k}[x_i], \mathbf{k}[x_i] \cdot h_i)^{Z_{h_i}(\Gamma)}. \tag{C.1}$$

Since $\mathbf{k}[x]$ has Hochschild dimension one (it has a projective bimodule resolution of length one), conclude that $\mathsf{HH}^j(\mathbf{k}[x], \mathbf{k}[x] \cdot h) = 0$ unless $j \le 1$. Using the explicit description as center and outer derivations of the module, show that, if $h \in \mathsf{GL}_1$ is not the identity,

$$\mathsf{HH}^0(\mathbf{k}[x], \mathbf{k}[x] \cdot h) = 0, \quad \mathsf{HH}^1(\mathbf{k}[x], \mathbf{k}[x] \cdot h) = \mathbf{k}.$$

Note for the second equality that you must remember to mod by inner derivations.
On the other hand, recall that

$$\mathsf{HH}^0(\mathbf{k}[x], \mathbf{k}[x]) = \mathbf{k}[x], \mathsf{HH}^1(\mathbf{k}[x], \mathbf{k}[x]) = \mathbf{k}[x],$$

since $\mathsf{HH}^\bullet(\mathbf{k}[x]) = \wedge^\bullet_{\mathbf{k}[x]} \operatorname{Der}(\mathbf{k}[x])$.
Now suppose in (C.1) that $h_i \neq \mathrm{Id}$ for $1 \le i \le j$, and that $h_i = \mathrm{Id}$ for $i > j$ (otherwise we can conjugate everything by a permutation matrix). Conclude that (C.1) implies

$$\mathsf{HH}^\bullet(A, A \cdot h)^{Z_h(\Gamma)} \cong \left(\mathsf{Sym}(V^h) \otimes (\partial_{x_1} \wedge \cdots \wedge \partial_{x_m}) \otimes \wedge_{\mathsf{Sym}(V^h)} \operatorname{Der}(\mathsf{Sym}(V^h)) \right)^{Z_h(\Gamma)}. \tag{C.2}$$

Note that, without having to reorder the x_i, we could write, for $(V^h)^\perp \subseteq V^*$ the subspace annihilating V^h,

$$\partial_{x_1} \wedge \cdots \wedge \partial_{x_m} = \wedge^{\dim (V^h)^\perp} \langle \partial_\phi \rangle_{\phi \in (V^h)^\perp}.$$

Put together, we get the statement. A similar argument works for Hochschild homology.

(d) Use the same method to prove the main result of [AFLS00] for V symplectic and $G < \mathsf{Sp}(V)$ finite:

$$\mathsf{HH}^i(\mathsf{Weyl}(V) \rtimes G) \cong \mathbf{k}[S_i]^G, \quad \mathsf{HH}_i(Weyl(V) \rtimes G) \cong \mathbf{k}[S_{\dim V - i}]^G$$

where

$$S_i := \{g \in \Gamma \mid \operatorname{rk}(g - \mathrm{Id}) = i.$$

In particular, explain why $S_i = \emptyset$ if i is odd, and $S_2 =$ the set of symplectic reflections (from Bellamy's lectures).

Hint: Recall from my lecture two that, for the quantization $\mathsf{Weyl}_\hbar(V)$ of $\mathcal{O}_{V^*}$, one has $\mathsf{HH}^\bullet(\mathsf{Weyl}_\hbar(V)[\hbar^{-1}]) \cong H^\bullet_{DR}(V^*, \mathbf{k}((\hbar))) \cong \mathbf{k}$. Then apply the result of part (b) and the method of part (c).

(3) Prove the proposition from the notes:

Proposition C.3. If $\mathsf{HH}^3(A) = 0$, then there exists a versal formal deformation of A with base $\mathbf{k}[\![\mathsf{HH}^2(A)]\!]$. If, furthermore, $\mathsf{HH}^1(A) = 0$, then this is a universal deformation.

In the case when A is filtered, then if $\mathsf{HH}^3(A)_{\leq 0} = 0$, there exists a versal filtered deformation of A with base $\mathbf{k}[HH^2(A)_{\leq 0}]$. If, furthermore, $\mathsf{HH}^1(A)_{\leq 0} = 0$, then this is a universal filtered deformation.

Hint: Use the Maurer-Cartan formalism, and the fact that, if $C^\bullet$ is an arbitrary complex of vector spaces, there exists a homotopy $H : C^\bullet \to C^{\bullet-1}$ such that $C^\bullet \cong H^\bullet(C) \oplus (hd + dh)(C^\bullet(C))$, i.e., $\mathrm{Id} - (Hd + dH)$ is a projection of $C^\bullet$ onto a subspace isomorphic to its homology. In this case, let $i : H^\bullet(C) \hookrightarrow C^\bullet$ be the obtained inclusion.

In the case $\mathsf{HH}^3(A) = 0$, let h be a homotopy as above for $C(A)$ and $i : \mathsf{HH}^\bullet(A) \to C^\bullet(A)$ the obtained inclusion. Show that a formula for a versal deformation $(A[\![\mathsf{HH}^2(A)]\!], \star)$, in the case $\mathsf{HH}^3(A) = 0$, can be given by the formal function $\Gamma \in C(A)[\![\mathsf{HH}^2(A)]\!]$,

$$\Gamma := (\mathrm{Id} + H \circ MC)^{-1} \circ i,$$

where $MC(x) := dx + \frac{1}{2}[x, x]$ is the LHS of the Maurer-Cartan equation. More precisely, plugging in any power series $\sum_{m\geq 1} \hbar^m \cdot \gamma_m \in \hbar \cdot \mathsf{HH}^2(A)[\![\hbar]\!]$, one obtains the Maurer-Cartan element $\sum_{m\geq 1} \hbar^m \Gamma(\gamma_m) \in \hbar \cdot C^2(A)[\![\hbar]\!]$, and the associated star products yield all possible formal deformations of A up to gauge equivalence.

(4) Prove that, if N is an A-bimodule, then A-bimodule derivations of N are the same as square-zero algebra extensions $A \oplus N$, i.e., algebra structures $(A \oplus N, \star)$ such that $N \star N = 0$, the bimodule action by $\star$ of A on N is the given one, and $a \star b \equiv ab \mod N$. Similarly:

(i) If B is a commutative algebra and N a B-module, show that commutative algebra derivations $\mathrm{Der}(B, N)$ are the same as square-zero commutative algebra extensions $B \oplus N$ of B.

(ii) If $\mathfrak{g}$ is a Lie algebra and N a $\mathfrak{g}$-module, show that Lie algebra derivations $\mathrm{Der}(\mathfrak{g}, N)$ are the same as square-zero Lie algebra extensions $\mathfrak{g} \oplus N$ of $\mathfrak{g}$.

(5) Prove the following:

The circle product also defines a natural structure on $\mathfrak{g} := C(A)[1]$, that of a dg *right pre-Lie* algebra: it satisfies the graded pre-Lie identity

$$\gamma \circ (\eta \circ \theta) - (\gamma \circ \eta) \circ \theta = (-1)^{|\theta||\eta|}\big(\gamma \circ (\theta \circ \eta) - (\gamma \circ \theta) \circ \eta\big),$$

as well as the compatibility with the differential,

$$d(\gamma \circ \eta) = (d\gamma) \circ \eta + (-1)^{|\gamma|}\gamma \circ (d\eta).$$

Given any dg (right) pre-Lie algebra, the obtained bracket

$$[x, y] = x \circ y - (-1)^{|x||y|} y \circ x$$

defines a dg Lie algebra structure.

(6) Exercise 4.23. Here is an expanded version:

For flat connections on the trivial principal G-bundle on a variety or manifold X, for G a Lie or algebraic group with $\mathfrak{g} = \mathrm{Lie}\, G$, gauge equivalence is defined as follows. Let

$\gamma : X \to G$ be a map, and $\gamma^{-1} : X \to G$ the composition with the inversion on G. Then γ defines the map on flat connections:

$$\nabla \mapsto (\operatorname{Ad}\gamma)(\nabla); (d+\alpha) \mapsto d + (\operatorname{Ad}\gamma)(\alpha) + \gamma \cdot d(\gamma^{-1}),$$

and these are the gauge equivalences.

Prove that, in the case that $\gamma = \exp(\beta)$ for $\beta \in \mathcal{O}_X \otimes \mathfrak{g}$, we can rewrite this as

$$\alpha \mapsto \exp(\operatorname{ad}\beta)(\alpha) + \frac{1-\exp(\operatorname{ad}\beta)}{\beta}(d\beta).$$

The last term should be thought of as $\exp(\operatorname{ad}\beta)(d)$, where we set $[d,\beta] = d(\beta)$.

To verify the formula, show that $\gamma \cdot d(\gamma^{-1}) = \sum_{m\geq 0} \frac{1}{m!}(\operatorname{ad}\beta)^{m-1}(-d(\beta))$. Use that $\operatorname{Ad}(\exp(\beta)) = \exp(\operatorname{ad}\beta)$ (this is true for all $\beta \in \mathfrak{g}^0$, and more generally for Lie algebras of connected Lie groups whenever $\exp(\beta)$ makes sense).

Appendix D. Exercise Sheet Four

Additional exercise: Exercise 4.10: verify Remark 4.9.

(1) Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Equip $\mathcal{O}_{\mathfrak{g}^*}$ with its standard Poisson structure. Show that, for Kontsevich's star product $\star$ on $\mathsf{Sym}\,\mathfrak{g}$ associated to this Poisson structure (use the description with graphs from the lecture!), one has

$$v \star w - w \star v = \hbar\{v,w\}, \quad \forall v,w \in \mathfrak{g}.$$

Hint: Show that only the graph corresponding to the Poisson bracket can give a nonzero contribution to $v \star w - w \star v$ when $v, w \in \mathfrak{g}$.

(2) If you haven't done it already, do Exercise Sheet 1 problem 6d.

Reprinted from the newly revised version of Exercise Sheet 1 (problem 6d):

Conclude from the final assertion of Exercise Sheet 1, problem 6.(c) that $\mathsf{HH}^\bullet(U\mathfrak{g}) = H^\bullet_{CE}(\mathfrak{g}, U\mathfrak{g}^{\mathrm{ad}})$. Namely, this assertion is (cf. [Lod98, Theorem 3.3.2]): if M is a $U\mathfrak{g}$-bimodule,

$$\mathsf{HH}^\bullet(U\mathfrak{g}, M) \cong H^\bullet_{CE}(\mathfrak{g}, M^{\mathrm{ad}}), \quad \mathsf{HH}_\bullet(U\mathfrak{g}, M) \cong H^{CE}_\bullet(\mathfrak{g}, M^{\mathrm{ad}}),$$

where M^{ad} is the $\mathfrak{g}$-module obtained from M by the adjoint action,

$$\operatorname{ad}(x)(m) := xm - mx,$$

where the LHS gives the Lie action of x on m, and the RHS uses the $U\mathfrak{g}$-bimodule action.

Next, recall (or accept as a black box) that, when $\mathfrak{g}$ is finite-dimensional semisimple (or more generally reductive), then $U\mathfrak{g}^{\mathrm{ad}}$ decomposes into a direct sum of finite-dimensional irreducible representations, and $H^\bullet_{CE}(\mathfrak{g}, V) = \operatorname{Ext}^\bullet_{U\mathfrak{g}}(\mathbf{k}, V) = 0$ if V is finite-dimensional irreducible and *not* the trivial representation.

Conclude that $\mathsf{HH}^\bullet(U\mathfrak{g}) = \operatorname{Ext}^\bullet_{U\mathfrak{g}}(\mathbf{k}, U\mathfrak{g}^{\mathrm{ad}}) = Z(U\mathfrak{g}) \otimes \operatorname{Ext}^\bullet_{U\mathfrak{g}}(\mathbf{k},\mathbf{k})$.

Bonus: Compute that $\operatorname{Ext}_{U\mathfrak{g}}(\mathbf{k},\mathbf{k}) = H^\bullet_{CE}(\mathfrak{g},\mathbf{k})$ equals $(\wedge^\bullet \mathfrak{g}^*)^{\mathfrak{g}}$. To do so, show that the inclusion $C^\bullet_{CE}(\mathfrak{g},\mathbf{k})^{\mathfrak{g}} \hookrightarrow C^\bullet_{CE}(\mathfrak{g},\mathbf{k})$ is a quasi-isomorphism. This follows by defining the oeprator d^* with opposite degree of d, so that the Laplacian $dd^* + d^*d$ is the quadratic Casimir element $C = \sum_i e_i f_i + f_i e_i + \frac{1}{2}h_i^2$. Thus this Laplacian operator defines a contracting homotopy onto the part of the complex on which the quadratic Casimir acts by zero. Since C acts by a positive scalar on all nontrivial finite-dimensional irreducible representations, this subcomplex is $C^\bullet_{CE}(\mathfrak{g},\mathbf{k})^{\mathfrak{g}}$. But the latter is just $(\wedge \mathfrak{g}^*)^{\mathfrak{g}}$ with zero differential.

Conclude that $\mathsf{HH}^\bullet(U\mathfrak{g}) = Z(U\mathfrak{g}) \otimes (\wedge^\bullet \mathfrak{g}^*)^{\mathfrak{g}}$. In particular, $\mathsf{HH}^2(U\mathfrak{g}) = \mathsf{HH}^1(U\mathfrak{g}) = 0$, so $U\mathfrak{g}$ has no nontrivial formal deformations.

Remark D.1. For $\mathfrak{g} = \mathfrak{sl}_2$, $U\mathfrak{sl}_2$ is a Calabi-Yau algebra, since it is a Calabi-Yau deformation of the Calabi-Yau algebra $\mathcal{O}_{\mathfrak{sl}_2^*} \cong \mathbf{A}^3$, which is Calabi-Yau with the usual volume form. So $\mathsf{HH}^3(U\mathfrak{sl}_2) \cong \mathsf{HH}_0(U\mathfrak{sl}_2)$ by Van den Bergh duality, and the latter is $U\mathfrak{sl}_2/[U\mathfrak{sl}_2, U\mathfrak{sl}_2]$, which you should be able to prove equals $(U\mathfrak{sl}_2)_{\mathfrak{sl}_2} \cong (U\mathfrak{sl}_2)^{\mathfrak{sl}_2}$, i.e., this is a rank-one free module over $\mathsf{HH}^0(U\mathfrak{sl}_2)$.

In other words, $H^3_{CE}(\mathfrak{g}, \mathbf{k}) \cong \mathbf{k}$, which is also true for a general semisimple Lie algebra. Since this group controls the *tensor category deformations* of $\mathfrak{g}$-mod, or the Hopf algebra deformations of $U\mathfrak{g}$, this is saying that there is a one-parameter deformation of $U\mathfrak{g}$ as a Hopf algebram which gives the quantum group $U_q\mathfrak{g}$. But since $\mathsf{HH}^2(U\mathfrak{g}) = 0$, as an associative algebra, this deformation is trivial, i.e., equivalent to the original algebra; this is *not* true as a Hopf algebra !

(3) Exercise 5.2.

Appendix E. Exercise Sheet Five

(1) Look at §2.3 of the notes.

Note that the description of §2.3 generalizes in an obvious way to define polydifferential operators

$$B_\Gamma(\gamma) : \mathcal{O}_X^{\otimes_{\mathbf{k}} m} \to \mathcal{O}_X, \quad \gamma : V_\Gamma \to T_{\text{poly}}(X),$$

where Γ is an arbitrary graph in the upper-half plane with m vertices on the real line, and $\gamma : V_\Gamma \to T_{\text{poly}}(X)$ is a function sending each vertex v to an element of degree equal to one less than the number of outgoing edges from v. Define this generalization, or alternatively see [Kon03].

Now, let $X = \mathfrak{g}^*$ equipped with its standard Poisson structure, for $\mathfrak{g}$ a finite-dimensional Lie algebra. Recall the wheels from §4.14. Let W_m denote the wheel with m vertices in the upper-half plane. We are interested in the operators

$$B_{W_m}(\pi, \ldots, \pi),$$

where π is the Poisson bivector on $\mathfrak{g}^*$.

(i) Prove that this operator is a constant-coefficient differential operator on $\mathfrak{g}^*$, i.e., an element of $\mathsf{Sym}\,\mathfrak{g}^*$. Each operator in $\mathsf{Sym}\,\mathfrak{g}^*$ is viewed as constant coefficient linear differential operator on $\mathcal{O}_{\mathfrak{g}^*} = \mathsf{Sym}\,\mathfrak{g}$, via the inclusion

$$\mathsf{Sym}^m\,\mathfrak{g}^* \hookrightarrow \mathrm{Hom}_{\mathbf{k}}(\mathsf{Sym}^\bullet\,\mathfrak{g}, \mathsf{Sym}^{\bullet - m}\,\mathfrak{g}).$$

In other words, if $\mathfrak{g}$ has a basis x_i with dual basis $\partial_i \in \mathfrak{g}^*$, then an element of $\mathsf{Sym}\,\mathfrak{g}^*$ is a polynomial in the ∂_i, i.e., a constant-coefficient differential operator.

Show, in fact, that $B_{W_m}(\pi, \ldots, \pi)$ is a differential operator of order m, i.e., that

$$B_{W_m}(\pi, \ldots, \pi) \in \mathsf{Sym}^m\,\mathfrak{g}^*.$$

(ii) In terms of $\mathsf{Sym}^m\,\mathfrak{g}^* = \mathcal{O}_{\mathfrak{g}}$, show that B_{W_m} is nothing but the polynomial function

$$x \mapsto \mathrm{tr}(\mathrm{ad}\,x)^m, x \in \mathfrak{g}.$$

(iii) Now suppose that Γ is a graph which consists of a single vertex on the x-axis, together with k wheels with $m_1, \ldots, m_k$ vertices in the upper half-plane, i.e., the union of the graphs $W_{m_1}, \ldots, W_{m_k}$ identifying all vertices on the x-axis to a single vertex. Show that, considered as polynomials in $\mathcal{O}_{\mathfrak{g}} = \mathsf{Sym}^m\,\mathfrak{g}^*$,

$$B_\Gamma(\pi, \ldots, \pi) = x \mapsto \prod_{i=1}^k \mathrm{tr}((\mathrm{ad}\,x)^{m_i}).$$

(iv) Conclude that Kontsevich's isomorphism

$$(\mathsf{Sym}\,\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} Z(\mathcal{O}_{\mathfrak{g}^*}[\![\hbar]\!], \star) \tag{E.1}$$

has the form, for some constants $c_{m_1,\ldots,m_k}$, now viewed as an element of $\hat{S}\mathfrak{g}^*$, i.e., a power series function contained in the completion $\hat{\mathcal{O}}_{\mathfrak{g}}$,

$$x \mapsto \sum_{k;m_1,\ldots,m_k} c_{m_1,\ldots,m_k} \cdot \prod_{i=1}^{k} \operatorname{tr}((\operatorname{ad} x)^{m_i}),$$

viewed as an element of $\hat{S}\mathfrak{g}^*$, i.e., a formal sum of differential operators (with finitely many summands of each order).

(v) Now, it follows from Kontsevich's explicit definition of the weights c_Γ associated to graphs Γ, which are the coefficients of B_Γ in the definition of his L_∞ quasi-isomorphism $\mathcal{U}$, that

$$c_{m_1,\ldots,m_k} = \prod_{i=1}^{k} c_{m_i}, \text{ and} \tag{E.2}$$

$$c_m = 0 \text{ if } m \text{ is odd.} \tag{E.3}$$

Using these identities (if you want to see why they are true, see [Kon03]), conclude that (E.1) is given by the (completed) differential operator corresponding to the series (cf. [Kon03, Theorem 8.5])

$$x \mapsto \exp\Big(\sum_{m\geq 1} c_{2m} \operatorname{tr}((\operatorname{ad} x)^{2m})\Big) \in \hat{S}\mathfrak{g}^* = \hat{\mathcal{O}}_{\mathfrak{g}}.$$

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