Normality and quadraticity for special ample line bundles on toric varieties arising from root systems (1)

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Oda's and Sturmfels' conjectures (2)

Conjecture (Oda)

Let V be a smooth, projective toric variety, and $\mathcal{L}$ an ample line bundle. Then, the embedding of V given by $\mathcal{L}$ is projectively normal, i.e., the map

$$H^0(V,\mathcal{L})^{\otimes k} \to H^0(V,\mathcal{L}^k) \tag{0.2}$$

is surjective for all $k \geq 1$.

In other words, the homogeneous coordinate ring of $V \subseteq \mathbf{P}(H^0(V,\mathcal{L}))$ is integrally closed, i.e., the affine cone over V is normal. (This depends on the embedding!!)

Conjecture (Sturmfels)

Additionally, the ring $\bigoplus_{k\geq 0} H^0(V,\mathcal{L}^{\otimes k})$ is quadratic.

In other words, Oda says that this ring is generated in degree $k = 1$, and Sturmfels says the relations are in degree $k = 2$.

Toric varieties associated to the Weyl fan (3)

Fix a connected, reductive, complex algebraic group G with maximal torus T and Borel B. We are interested in the variety V obtained as the closure of a generic T-orbit in the flag variety G/B.

This variety V has been studied by many authors since at least 1990. It is a toric variety:

Let Δ be the set of roots, and let $Y, X, X^\vee$, and $Y^\vee$ be the root, weight, coroot, and coweight lattices, so $Y \subseteq X$ and $X^\vee \subseteq Y^\vee$.

Let $F \subseteq Y^\vee \otimes_{\mathbf{Z}} \mathbf{R}$ be the Weyl fan: this is the fan whose maximal facets are the Weyl chambers.

Goal: to prove Oda's and Sturmfels' conjecture for certain ample line bundles on this variety V.

From now on V is as above and $T \subseteq V$ is the torus.

Special ample line bundles (4)

T-equivariant ample line bundles $\mathcal{L}$ on V are equivalent to polytopes $P \subseteq Y \otimes_{\mathbf{Z}} \mathbf{R}$ whose vertices μ_σ are in bijection with the Weyl chambers σ, such that, for adjacent chambers σ and σ',

$$\mu_\sigma - \mu_{\sigma'} = r_{\sigma,\sigma'}\alpha_{\sigma,\sigma'}, r_{\sigma,\sigma'} > 0,$$

where $\alpha_{\sigma,\sigma'}$ is the unique root positive on σ and negative on σ'. A basis of T-stable vectors of $H^0(V, \mathcal{L})$ is given by the lattice vectors $Y \cap P$ of P.

Definition

(e.g., Kottwitz) A *special ample line bundle* on V is one such that μ_σ lies in the Weyl chamber corresponding to σ, for all σ. Call P a *special ample polytope*.

Theorem (Gashi, S.)

Let $\mathcal{L}$ be a special ample line bundle on V. Then, the homogeneous coordinate ring $H^0(V, \mathcal{L}^k)$ is normal and quadratic.

The key proposition and a lemma of Stembridge (5)

Let P be a polytope corresponding to a special ample line bundle. Think of this as a convex region in $X \otimes_{\mathbf{Z}} \mathbf{R}$. For any root α, let $\alpha^\vee$ be the corresponding coroot.

Proposition

If $v \in P$ and $\langle v, \alpha^\vee \rangle \geq 1$ for $\alpha \in \Delta$, then $v - \alpha \in P$.

Next, for two dominant weights $x, y \in Y$, say that $x \preceq y$ if $y - x$ is a nonnegative combination of positive roots. Say that y *covers* x if $x \prec y$ and, for all $z \in Y$ such that $x \preceq z \preceq y$, either $z = x$ or $z = y$.

Lemma (Stembridge)

Suppose that x, y are dominant and y covers x. Then $y - x \in \Delta$.

Proof of normality (6)

We have to show that, for all $k \geq 1$ and all lattice vectors $x \in kP \cap X$ (corresponding to a basis vector of $H^0(V, \mathcal{L}^k)$), that there exist lattice vectors $x_1, \ldots, x_k \in P \cap X$ such that $x = x_1 + \cdots + x_k$.

Using the action of the Weyl group (which sends P to another special ample polytope), it is enough to suppose that x is dominant. Let σ be the dominant chamber.

We proceed by induction on the order $\prec$. The top such in kP is the vertex $k\mu_\sigma = \mu_\sigma + \cdots + \mu_\sigma$.

All other $x \in Y \cap kP$ satisfy $x \preceq k\mu_\sigma$. So it is enough to show that: if $x, y \in Y \cap kP \cap \sigma$, y covers x, and $y = y_1 + \cdots y_k$ for $y_i \in Y \cap P$, then also $x = x_1 + \cdots + x_k$ for some $x_i \in Y \cap P$.

Indeed, by Stembridge's lemma, $x = y - \alpha$ for a positive root α. Since x and y are dominant, $\langle y, \alpha^\vee \rangle \geq 2$. So, for some i, $\langle y_i, \alpha^\vee \rangle \geq 1$.

Then the proposition implies that $y_i - \alpha \in P$ as well. So, $x = y_1 + \cdots + y_{i-1} + (y_i - \alpha) + y_{i+1} + \cdots + y_k$. $\square$

Proof of quadraticity (7)

Above we used that, if P is special ample, so is kP. More generally, if $P_1, \ldots, P_k$ are special ample, so is $P_1 + \cdots + P_k$. The following generalizes our main theorem:

Theorem (Gashi-S.)

(i) *If $x \in Y \cap (P_1 + \cdots + P_k)$ then $x = x_1 + \cdots + x_k$ for some $x_i \in Y \cap P_i$.*

(ii) *If $x_1 + \cdots + x_k = x'_1 + \cdots x'_k$ for $x_i, x'_i \in Y \cap P_i$, then $(x_1, \ldots, x_k)$ and $(x'_1, \ldots, x'_k)$ are related by moves ($\alpha \in \Delta$):*

$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{i-1}, x_i + \alpha, x_{i+1}, \ldots, x_{j-1}, x_j - \alpha, x_{j+1}, \ldots, x_k).$$

Now, we can induct on k, applying the pair $(P_1, P_2 + \cdots + P_k)$. This reduces to the case $k = 2$. We will only prove (ii), since (i) is essentially the same as before.

Strengthened normality and quadraticity

In fact the strengthened theorem implies:

Corollary

Let $\mathcal{L}_1, \ldots, \mathcal{L}_k$ be special ample line bundles. Then the multi-homogeneous coordinate ring $\bigoplus_{m \geq 0} H^0(V, \mathrm{Sym}^m(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_k)))$ is normal and quadratic.

Remark

In terms of polytopes, this says that the Cayley sum polytope $P_1 * \cdots * P_k$ is normal and its semigroup ring is quadratic. [[This polytope is the convex hull of $\bigcup P_i \times \{e_i\} \subseteq (Y \otimes_{\mathbf{Z}} \mathbf{R}) \times \mathbf{R}^k$ where $\mathbf{R}^k$ has basis vectors $e_1, \ldots, e_k$.]]

Quadraticity for $k = 2$ (9)

Suppose $x_1 + x_2 = x = x_1' + x_2'$ for $x_i, x_i' \in P_i$. We may assume x is dominant and induct on $\prec$.

Lemma

Let α be a simple root, $(x_1, x_2), (x_1 + \alpha, x_2) \in P_1 \times P_2$, and $\langle x, \alpha^\vee \rangle \geq -1$. If $((x_1 + \alpha) + \beta, x_2 - \beta) \in P_1 \times P_2$ for $\beta \in \Delta$, then:

(i) *$(x_1 + \beta, x_2 - \beta) \in P_1 \times P_2$; or*

(ii) *$(x_1 + (\alpha + \beta), x_2 - (\alpha + \beta)) \in P_1 \times P_2$, and either: (a) $\alpha + \beta \in \Delta$, or (b) $(x_1 + \alpha, x_2 - \alpha) \in P_1 \times P_2$.*

Using the lemma, provided the theorem (ii) holds for $y = x + \alpha$ with $\langle y, \alpha \rangle \geq 1$ and α *simple*, it also holds for x.

Strengthening of Stembridge's lemma (10)

Problem: if $x \in P := P_1 + P_2$ is dominant and $x \prec \mu$, there need not exist a simple α with $x + \alpha$ dominant and in P.

Lemma

Suppose that $x \in P$ is dominant, $x \prec \mu$ and α is simple. Then there exists a sequence $x + \alpha = u_1 \mapsto u_2 \mapsto \cdots \mapsto u_m = y$ s.t.:

(i) *$x \prec y \preceq \mu$, and y is dominant and covers x;*

(ii) *All the u_i are in P;*

(iii) *For all i, $u_{i+1} = u_i + \beta_i$ for β_i simple and $\langle u_i, \beta_i^\vee \rangle = -1$.*

In fact, (iii) implies (i) and (ii) using the main proposition ($u_i \in P$ if and only if $u_{i+1} \in P$ since $\langle u_{i+1}, \alpha^\vee \rangle \geq 1$ and $\langle u_i, \alpha^\vee \rangle \leq -1$).
To prove (iii), we use a study of *the numbers game*, of which the above are moves.
Now, inductively, we can deduce the result for $x \in Y \cap P \cap \sigma$ from the result for $y \in Y \cap P \cap \sigma$. $\square$

Questions! (11)

1. **Koszulity:** Is the homogeneous coordinate ring $\bigoplus_{k\geq 0} H^0(V, \mathcal{L}^k)$ Koszul?
   Remark: Sam Payne studied a dual variety to V and, in types A, B, C, and D, he proved the corresponding ring is normal and Koszul (without speciality assumptions); his methods do not extend to exceptional type, and do not work here.
   (Also, what about extending Sturmfels' conjecture, replacing quadraticity with Koszulity? There are no known counterexamples.)
2. **Speciality:** Is there a way to drop the speciality assumption on the bundle $\mathcal{L}$?
3. **White's Conjecture:** This says that semigroup rings associated to matroid polytopes are quadratic. In the type A case, these correspond to certain nef divisors which are not ample (except for the uniform matroid). Can our methods extend to this case? (Question posed by Sam Payne).