Lecture 18: Normal operators, the spectral theorems, isometries, and positive operators (1)

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Tue, Nov 16, 2010 (version: Tue, Nov 16, 4:00 PM)

Goals (2)

• Normal operators and the spectral theorem
  – Nice corollaries (slides (7)–(10)) which I plan to skip, but you should study!
• Isometries
• Positive operators
• Polar decomposition

Spectral theorem for self-adjoint operators (3)

From now on, all our vector spaces are finite-dimensional inner product spaces.

Theorem 1 (Theorem 7.13+). $T$ is self-adjoint iff $T$ admits an orthonormal eigenbasis with real eigenvalues.

Proof. • Proof for $F = \mathbb{C}$: we already know that $M(T)$ is upper-triangular in some orthonormal basis.

• Then, $T = T^*$ iff the matrix equals its conjugate transpose, i.e., it is upper-triangular with real values on the diagonal.

• Now let $F = \mathbb{R}$. In some orthonormal basis, the matrix is block upper-triangular with $1 \times 1$ and $2 \times 2$ blocks.

• Then, the matrix equals its own transpose iff it is block diagonal with real diagonal entries and symmetric $2 \times 2$ blocks.

• However, in slide (6) we show that the $2 \times 2$ blocks are anti-symmetric. So there are none. 

\[\square\]
Spectral theorem for complex normal operators (4)

Motivation: Which $T$ admit an orthonormal eigenbasis but not necessarily with real eigenvalues?

**Definition 2.** An operator $T$ is normal if $TT^* = T^*T$, i.e., $T$ and $T^*$ commute.

**Theorem 3** (Theorem 7.9). Let $F = \mathbb{C}$. Then $T \in \mathcal{L}(V)$ admits an orthonormal eigenbasis iff it is normal.

**Proof.**

- Pick an orthonormal basis so that $A := \mathcal{M}(T)$ is upper-triangular.

  Then $T$ is normal iff $A A^t = A^t A$.

- In coordinates $A = (a_{jk})$ (with $a_{jk} = 0$ for $j > k$), this means $|a_{jj}|^2 + \cdots + |a_{jn}|^2 = |a_{jj}|^2$, $\forall j$. (dim $V = n$)

- This is equivalent to: $a_{jk} = 0$ for $j < k$. So $T$ is normal iff $A$ is diagonal.

\[\square\]

Spectral theorem for real normal operators (5)

Motivation: What does it mean for a real operator to be normal?

**Theorem 4** (Theorem 7.25). Let $F = \mathbb{R}$. Then $T$ is normal iff it admits an orthonormal basis in which $\mathcal{M}(T)$ is block-diagonal with blocks $(\lambda_j)$ or $\begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}$.

The complex eigenvalues are $\lambda_j \in \mathbb{R}$ and $a_j \pm i b_j$.

**Proof.**

- Pick an orthonormal basis so that $A = \mathcal{M}(T)$ is block upper-triangular.

  Then, $T$ is normal iff $A A^t = A^t A$.

- For the rows with $1 \times 1$ blocks, this again means $|a_{jj}|^2 + \cdots + |a_{jn}|^2 = |a_{jj}|^2$, $\forall j$.

- For rows $j, j + 1$ with $2 \times 2$ blocks, adding the corresponding sums for both rows, this implies $\sum_{k=j+2}^{n} |a_{jk}|^2 + |a_{j+1,k}|^2 = 0$, $\forall j$, $a_{j,k} = a_{j+1,k} = 0$ for $k > j + 1$, so $A$ is block diagonal.

- Finally, we apply the following proposition to the blocks. \[\square\]

2 $\times$ 2 case (6)

We need just one final detail ($F = \mathbb{R}$ and dim $V = 2$):

**Proposition 0.1** (Lemma 7.15, essentially). Suppose that $T \in \mathcal{L}(V)$ is normal and that $T$ has no eigenvalues. Then, in any orthonormal basis, $\mathcal{M}(T) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $a \pm bi$ are the roots of the characteristic polynomial of $T$. 

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Recall that for two-by-two matrices $A$, the characteristic polynomial is $x^2 - (\text{tr} \, A)x + \det A$, and this does not depend on the choice of basis so makes sense for $T$.

**Proof.**

- Write $\mathcal{M}(T) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the orthonormal basis.
  
  - Since $T$ is normal, $|a|^2 + |b|^2 = |a|^2 + |c|^2$, so $b = \pm c$.
  
  - Since there are no real eigenvalues of $\mathcal{M}(T)$, $b = -c \neq 0$.
  
  - Since $T$ is normal, $ac + bd = ab + cd$, so $(d-a)b = (a-d)b$. So $a = d$. 

**Corollaries (7)**

**Corollary 5** (Corollary 7.8). If $T$ is normal, then eigenvectors $u, v$ of distinct eigenvalues are orthogonal.

- Proof: In an orthonormal eigenbasis $(e_j)$ so that $\mathcal{M}(T)$ is (block) upper-triangular, an eigenvector $v$ of eigenvalue $\lambda$ is a linear combination of the $e_j$ with the same eigenvalue.

- So $u, v$ cannot have nonzero coefficients of the same $e_j$, i.e., $u \perp v$.

**Corollary 6** (Corollary 7.7). Let $T$ be normal. If $v$ is an eigenvector of $T$ of eigenvalue $\lambda$, then it is also an eigenvector of $T^*$ of eigenvalue $\bar{\lambda}$.

- Proof: Again, $v$ must be a linear combination of the $e_j$ that are eigenvectors of eigenvalue $\lambda$.

- Since $\mathcal{M}(T^*) = \mathcal{M}(T)^t$, these $e_j$ are eigenvectors of $T^*$ of eigenvalue $\bar{\lambda}$.

**A characterization of normal operators (8)**

**Proposition 0.2** (Proposition 7.4). If $T$ is self-adjoint, then $\langle Tv, v \rangle = 0$ for all $v$ iff $T = 0$.

- Assume $\langle Tv, v \rangle = 0$ for all $v$. Then $\langle T(v+u), v+u \rangle - \langle T(v), v \rangle - \langle T(u), u \rangle = 0$ for all $v, u$.

- Thus, $\langle T(v), u \rangle + \langle T(u), v \rangle = 0$ for all $u, v$.

- When $T$ is self-adjoint, this says $2\Re \langle T(v), u \rangle = 0$ for all $u, v$.

- Plugging in $iu$ for $u$, also $2\Im \langle T(v), u \rangle = 0$ for all $u, v$. Thus $T(v) = 0$ for all $v$, i.e., $T = 0$.

**Corollary 7** (Proposition 7.6). An operator $T$ is normal iff $\|Tv\| = \|T^*v\|$ for all $v$.

- $\|Tv\| = \|T^*v\| \iff \langle Tv, Tv \rangle = \langle T^*v, T^*v \rangle \iff \langle (T^*T - TT^*)v, v \rangle = 0$.

- Since $T^*T - TT^*$ is self-adjoint, by the corollary, the last condition is satisfied for all $v$ iff $T$ is normal.
\[ \langle Tv, v \rangle = 0 \] and anti-self-adjoint operators on \( \mathbb{R} \) (9)

**Proposition 0.3** (Proposition 7.2+). \( \langle Tv, v \rangle = 0 \) for all \( v \) iff either \( T = 0 \), or \( F = \mathbb{R} \) and \( T = -T^* \) (\( T \) is anti-self-adjoint).

Note: anti-self-adjoint (\( T = -T^* \)) implies normal.

**Proof.**
- Take an orthonormal basis \((e_j)\) in which \( A = (a_{jk}) = \mathcal{M}(T) \) is (block) upper-triangular.
- We claim \( a_{jj} = 0 \) for all \( j \). Indeed, \( \langle Te_j, e_j \rangle = a_{jj} = 0 \).
- It remains only to show that \( A \) is block diagonal (since then the blocks are antisymmetric).
- Otherwise, if \( a_{jk} \neq 0 \) above the block diagonal, then \( \langle T(e_j + \lambda e_k), e_j + \lambda e_k \rangle = \langle (a_{jj} + \lambda a_{jk})e_j, e_j \rangle = a_{jj} + \lambda a_{jk} \). For \( \lambda \neq -a_{jj}^{-1}a_{jk} \), this is nonzero. Contradiction. \( \square \)

**Anti-self-adjoint operators for \( F = \mathbb{C} \)** (10)

**Proposition 0.4.** For \( F = \mathbb{C} \), \( T = -T^* \) if and only if \( T \) has an orthonormal eigenbasis with purely imaginary eigenvalues (i.e., eigenvalues in \( i \cdot \mathbb{R} \)).

**Proof.**
- In an orthonormal basis in which \( \mathcal{M}(T) \) is upper-triangular, \( T = -T^* \) means the matrix equals its negative conjugate transpose.
- This means it is diagonal with purely imaginary diagonal entries. \( \square \)

Alternatively: anti-self-adjoint operators are normal, so admit an orthonormal eigenbasis; then \( \langle Tv, v \rangle = \lambda \langle v, v \rangle = \langle v, T^*v \rangle = -\overline{\lambda} \langle v, v \rangle \) implies that \( \lambda = -\overline{\lambda} \) for all eigenvalues \( \lambda \). So they are purely imaginary.

**Complex eigenvalues of real operators** (11)

We have often spoken about complex eigenvalues of \( \mathcal{M}(T) \) when \( F = \mathbb{R} \). Let’s formalize it:

**Definition 8.** Let \( F = \mathbb{R} \). Then the **complex eigenvalues** of \( T \) are the complex eigenvalues of \( \mathcal{M}(T) \) in any basis.

Why do these not depend on the basis? The change of basis formula! We know that conjugate matrices \( A \) and \( SAS^{-1} \) have the same (complex) eigenvalues (directly, or by using complex operators).

**Example 9.** The complex eigenvalues of any \( T \) such that \( \mathcal{M}(T) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \) are \( a + bi \) and \( a - bi \).

- Note: \( |a + bi| = 1 = |a - bi| \) iff it is a rotation matrix, i.e., \( a = \cos \theta \) and \( b = \sin \theta \) for some angle \( \theta \).
Isometries (12)

Let $T \in \mathcal{L}(V, W)$, with $V$ and $W$ inner product spaces.

Definition 10. An isometry is an operator such that $\langle u, v \rangle = \langle Tu, Tv \rangle$ for all $u, v \in V$.

That is, isometries are operators that preserve the inner product.

Proposition 0.5. $T$ is an isometry iff it preserves merely the norm: $\|Tv\| = \|v\|$ for all $v$.

Proof. The inner product is given by a formula from the norm (on the homework), so preservation of the norm implies preservation of inner product. The converse is obvious. \qed

Characterization of isometries (13)

Let $V = W$ be finite-dimensional. Useful characterization: $T \in \mathcal{L}(V)$ is an isometry iff $T^*T = I = TT^*$. (In particular isometries of $V$ are invertible!)

Theorem 11 (Theorem 7.37). Isometries are the same as normal operators whose complex eigenvalues all have absolute value one.

Proof. 

• First, isometries are normal by the characterization.

• Given a normal operator, pick an orthonormal basis as in the spectral theorem. Then $\mathcal{M}(T)\mathcal{M}(T^*) = I$ iff $|\lambda_i|^2 = 1$ and $|a_i|^2 + |b_i|^2 = 1$ for all $i$. \qed

That is, in our usual orthonormal basis, an isometry has blocks which are either numbers of absolute value one, or rotation matrices.

Positive operators (14)

Definition 12. A positive operator $T$ is a self-adjoint operator such that $\langle Tv, v \rangle \geq 0$ for all $v$.

In view of the spectral theorem, a self-adjoint operator is positive iff its eigenvalues are nonnegative (part of Theorem 7.27).

Theorem 13 (Remainder of Theorem 7.27). (i) Every operator of the form $T = S^*S$ is positive.

(ii) Every positive operator admits a positive square root.

Proof. 

• (i) First, $T^* = (S^*S) = S^*S$ is self-adjoint.

• Next, $\langle Tv, v \rangle = \langle Sv, Sv \rangle \geq 0$ for all $v$.

• (ii) For any orthonormal eigenbasis of $T$, let $\sqrt{T}$ be the operator with the same orthonormal eigenbasis, but with the nonnegative square root of the eigenvalues. \qed
Polar decomposition (15)

After the spectral theorem, the second-most important theorem of Chapters 6 and 7 is:

**Theorem 14** (Polar decomposition: Theorem 7.41). Every $T \in \mathcal{L}(V)$ equals $S\sqrt{T^*T}$ for some isometry $S$.

Main difficulty: $T$ need not be invertible!

**Lemma 15.** For all $v \in V$, $|Tv| = \|\sqrt{T^*T}v\|$.  

**Proof.** $\|\sqrt{T^*T}v\|^2 = \langle \sqrt{T^*T}v, \sqrt{T^*T}v \rangle = \langle (\sqrt{T^*T})^*\sqrt{T^*T}v, v \rangle = \langle T^*Tv, v \rangle = \langle Tv,Tv \rangle = \|Tv\|^2$. □

Corollary: $\text{null}(T) = \text{null}(\sqrt{T^*T})$. We may thus define $S_1 : \text{range}(\sqrt{T^*T}) \to \text{range}(T) \ni S_1(\sqrt{T^*T}v) = Tv$. Thus, for all $v \in V$, $S_1\sqrt{T^*T}v = Tv$. Also, $\|S_1u\| = \|u\|$ for all $u(= \sqrt{T^*T}v)$ by the lemma. So $S_1$ is an isometry.

**Completion of proof (16)**

- We only have to extend $S_1$ to an isometry on all of $V$.
- Note that $\text{range}(\sqrt{T^*T}) \oplus \text{range}(\sqrt{T^*T})^\perp = V = \text{range}(T) \oplus \text{range}(T)^\perp$.
- Thus, the extensions of $S_1 : \text{range}(\sqrt{T^*T}) \to \text{range}(T)$ to an isometry $S : V \to V$ are exactly $S = S_1 \oplus S_2$, where $S_2 : \text{range}(\sqrt{T^*T})^\perp \to \text{range}(T)^\perp$ is an isometry.
- Since these are inner product spaces of the same dimension, there always exists an isometry, by taking an orthonormal basis to an orthonormal basis.

Recall here that $T_1 \oplus T_2$ on $U_1 \oplus U_2$ means $(T_1 \oplus T_2)(u_1 + u_2) = T_1(u_1) + T_2(u_2)$, $\forall u_1 \in U_1, u_2 \in U_2$. 
