On Decompositions, Partitions, and Coverings with Convex Polygons and Pseudo-Triangles

O. Aichholzer[*] C. Huemer[†] S. Kappes[‡] B. Speckmann[§] C. D. Tóth[¶]

Abstract

We propose a novel subdivision of the plane that consists of both convex polygons and pseudo-triangles. This *pseudo-convex* decomposition is significantly sparser than either convex decompositions or pseudo-triangulations for planar point sets and simple polygons. We also introduce pseudo-convex partitions and coverings. We establish some basic properties and give combinatorial bounds on their complexity. Our upper bounds depend on new Ramsey-type results concerning disjoint empty convex k-gons in point sets.

1 Introduction

Geometric algorithms and data structures frequently use subdivisions of the input space into compact and easy to handle polygonal cells. Triangulations are among the most widely used of these tessellations. Since the running time of algorithms is often correlated with the size of the subdivision, many efficient algorithms tile the plane with generalizations of triangles such as convex polygons or pseudo-triangles which provide a sparser tessellation but retain many of the desirable properties of a triangulation. Both convex subdivisions and pseudo-triangulations have applications in areas like motion planning [7, 30], collision detection [1, 21], ray shooting [6, 14], or visibility [25, 27, 26]. A pseudo-triangle is the "most reflex" polygon possible—it has exactly three convex vertices with internal angles less than π. Whether a chain of points is considered convex or reflex depends only on the point of view. So pseudo-triangles can be considered as natural counterparts of convex polygons.

In this paper we propose a combination of convex and pseudo-triangular subdivisions: *Pseudo-convex* decompositions. A pseudo-convex decomposition is a tiling of the plane with convex polygons and pseudo-triangles. We also introduce the related concepts of pseudo-convex partitions and coverings whose convex counterparts have been extensively studied as well. We establish some basic combinatorial properties and give quantitative bounds on the complexity of pseudo-convex decompositions, partitions, and coverings for point sets and simple polygons. Pseudo-convex decompositions are significantly sparser than convex decompositions or pseudo-triangulations.

All our bounds are combinatorial, in fact we do not know what the complexity of finding a minimum decomposition for a given input point set is. Our upper bounds depend on optimal solutions for small point configurations. Any improvement on a finite point set would lead to better bounds. We achieve optimal bounds for small configurations by proving two geometric Ramsey-type results concerning disjoint empty convex k-gons in point sets. These results extend previous work by Erdős, Hosono, and Urabe, but to the best of our knowledge our results are the first Ramsey-type answers to such questions. Small configurations of points are notoriously hard to deal with. An asymptotic lower bound for the number of order types of a set of n points in the plane is $n^{\Theta(n \log n)}$ [13]. We confirmed our conjectures regarding sets of 8 and 11 points with the help of the order type data base developed at TU Graz [2, 3]. We give analytical proofs for some of our results, while others are purely based on the data base.

Organization. The next paragraphs give precise definitions for convex and pseudo-convex decompositions, partitions, and coverings and Section 2 collects some of their basic combinatorial properties. In the next subsection we state our results and compare our bounds to previous work. Pseudo-convex decompositions and partitions are significantly sparser than their convex counterparts while pseudo-convex and convex coverings have asymptotically the same complexity. We devote Section 3 to pseudo-convex decompositions and Section 4 to pseudo-convex partitions of point sets. Subsection 3.1 formally states our two Ramsey-type theorems. Section 5 collects a number of observations concerning pseudo-convex coverings for small point sets. Finally, Section 6 discusses pseudo-convex decompositions for the interior of simple polygons. We conclude with some open problems.

[*]Institute for Software Technology, Graz University of Technology, oaich@ist.tugraz.at
[†]Departament de Matemática Aplicada II, Universitat Politécnica de Catalunya, huemer.clemens@upc.edu
[‡]Department of Mathematics, TU Berlin, kappes@math.TU-Berlin.de
[§]Department of Mathematics and Computer Science, TU Eindhoven, speckman@win.tue.nl
[¶]Department of Mathematics, Room 2-336, MIT, 77 Massachusetts Ave., Cambridege, MA 02144, toth@math.mit.edu

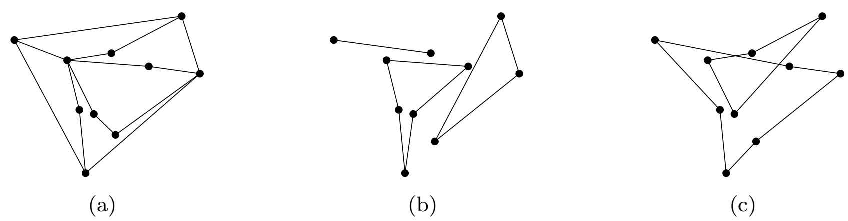


Figure 1: A pseudo-convex decomposition (a), a pseudo-convex partition (b), and a pseudo-convex covering (c).

Definitions. Let S be a set of n points in general position in the plane. A *pseudo-triangle* is a planar polygon that has exactly three convex vertices with internal angles less than π, all other vertices are concave. A *pseudo-triangulation* of S is a subdivision of the convex hull of S into pseudo-triangles whose vertex set is exactly S. A vertex is called *pointed* if it has an adjacent angle greater than π. A planar straight line graph is pointed if every vertex is pointed.

The *convex decomposition number* of S, $\kappa_d(S)$, is the minimum number of faces in a subdivision of the convex hull of S into convex polygons whose joint vertex set exactly S. A *pseudo-convex decomposition* of S is a subdivision of the convex hull of S into a family of convex polygons and pseudo-triangles whose joint vertex set is S. For instance every triangulation or pseudo-triangulation of S is a pseudo-convex decomposition. The *pseudo-convex decomposition number* of S, $\psi_d(S)$, is the minimum number of faces in a pseudo-convex decomposition of S.

The *convex partition number* of S, $\kappa_p(S)$, is the minimum number of pairwise disjoint closed convex polygonal domains whose joint vertex set is S (that is, the vertex sets of these convex polygons partition the point set S). Similarly, the *pseudo-convex partition number* of S, $\psi_p(S)$, is the minimum number of pairwise disjoint, closed, convex or pseudo-triangular polygonal domains whose joint vertex set is S. Note that each of the disjoint polygonal domains is empty (of points): neither a convex nor a pseudo-convex partition contains nested polygons.

The *convex cover number* of S, $\kappa_c(S)$, is the minimum number of convex polygons whose joint vertex set is S. Similarly, the *pseudo-convex cover number* of S, $\psi_c(S)$, is the minimum number of convex and pseudo-triangular polygons whose joint vertex set is S.

We are interested in combinatorial bounds on the maximum pseudo-convex decomposition (resp., pseudo-convex partition, pseudo-convex cover) number over all point sets of a given size $n \in \mathbb{N}$. We let

$$\psi_i(n) := \max_{|S|=n} \psi_i(S), \qquad \text{for } i = d, p, c.$$

Similarly, for the convex decompositions (resp., partitions and coverings), we let $\kappa_i(n) := \max_{|S|=n} \kappa_i(S)$, for $i = d, p, c$.

1.1 Previous work and results.

Decomposition. The convex decomposition number $\kappa_d(n)$ is bounded by

$$\frac{12}{11}n - 2 < \kappa_d(n) \le \frac{10n - 18}{7} .$$

The lower bound was given very recently by García-López and Nicolás [11] and the upper bound was established by Neumann-Lara et al. [24]. Fevens, Meijer, and Rappaport [10] and Spillner [29] designed algorithms for computing a minimum convex decomposition for input point sets. Every minimum pseudo-triangulation of n points has exactly $n-2$ pseudo-triangles [30]. We show that the pseudo-convex decomposition number is bounded by

$$\frac{3}{5}n \le \psi_d(n) \le \frac{7}{10}n .$$

Furthermore, we also prove that $\psi_d(n)$ is monotonically increasing with n.

Partition. The convex partition number $\kappa_p(n)$ is bounded by

$$\left\lceil \frac{n-1}{4} \right\rceil \le \kappa_p(n) \le \left\lceil \frac{5n}{18} \right\rceil .$$

The lower bound was given by Urabe [31] and the upper bound was established by Hosono and Urabe [16]. Arkin et al. [4] study questions related to convex partitions and coverings by examining the reflexivity of point sets. We show that the pseudo-convex partition number $\psi_p(n)$ is bounded by

$$\left\lfloor \frac{3n}{16} \right\rfloor \leq \psi_p(n) \leq \frac{n}{4} .$$

Covering. The study of convex cover numbers is rooted in the classical work of Erdős and Szekeres [8, 9] who showed that any set of n points contains a convex subset of size $\Omega(\log n)$. More recent results include the work by Urabe [31] who proved that the convex cover number $\kappa_c(n)$ is bounded by

$$\frac{n}{\log_2 n + 2} < \kappa_c(n) < \frac{2n}{\log_2 n - \log_2 e} .$$

There is an easy connection between the pseudo-convex cover number and the convex cover number, namely $\psi_c(n) \leq \kappa_c(n) \leq 3\psi_c(n)$ (all points which can be covered by a pseudo-triangle can be covered by at most three convex sets). Thus both numbers have the same asymptotic behavior, which implies

$$\psi_c(n) = \Theta\left(\frac{n}{\log n}\right) .$$

Geometric Ramsey-type Results. The upper bound construction for $\psi_d(n)$ relies on minimal pseudo-convex decomposition numbers for few points. These are, in turn, related to a combinatorial geometry problem on empty convex polygons that goes back to Erdős: For $k \geq 3$ find the smallest integer $E(k)$ such that any set S of $E(k)$ points contains the vertex set of a convex k-gon whose interior does not contain any point of S. Klein [8] showed that every set of 5 points contains an empty convex quadrilateral, that is $E(4) = 5$. Harborth [15] proved that every set of 10 points contains an empty convex pentagon, that is $E(5) = 10$. In the last decade, Urabe [31] proved that every set of 7 points can be partitioned into a triangle and a disjoint convex quadrilateral. Hosono and Urabe [16] showed that every set of 9 points contains two disjoint empty convex quadrilaterals. Very recently Gerken showed that any set that contains a convex 9-gon also contains an empty convex hexagon. Each of these results corresponds to a bound on the pseudo-convex decomposition number $\psi_d(n)$. The best upper bound we achieved depends on new results for empty convex polygons.

A typical Ramsey type problem asks for the minimum size of a system that contains at least one of two (or more) subconfigurations. The classical Ramsey number $R(n, m)$ is the smallest integer such that every red-blue complete graph on $R(n, m)$ vertices contains a red K_n or a blue K_m. The first geometric Ramsey-type problems focused on geometric graphs [17, 18] and intersection graphs [22].

We prove the following two results: (1) Every set of 8 points in general position contains either an empty convex pentagon or two disjoint empty convex quadrilaterals. (2) Every set of 11 points in general position contains either an empty convex hexagon or an empty convex pentagon and a disjoint empty convex quadrilateral.

Simple Polygons. An initial step of many algorithms on simple polygons is a decomposition into simpler components [19]. Keil and Snoeyink [20] devised an algorithm for computing the minimum convex decomposition of the interior of a given simple polygon. Chazelle and Dobkin [5] studied a variant of this optimization problem allowing Steiner points, Lien and Amato [23] constructed approximately convex decompositions. Motivated by our early results, obtained during our investigations for this paper, Gerdijkov and Wolff [12] extended the work by Keil and Snoeyink to compute the minimum pseudo-convex decomposition of a simple polygon.

The minimum convex decomposition of a pseudo-triangle with n vertices may require $n - 2$ triangles, and the minimum pseudo-triangulation of any convex n-gon is a triangulation with $n - 2$ faces. (In these extremal examples, Steiner points do not lead to a smaller convex decomposition or pseudo-triangulation.) We show that any n-gon has a pseudo-convex decomposition of size $\lceil n/2 \rceil - 1$.

Note that any quadrangulation (a decomposition into quadrilaterals) of an n-gon is a pseudo-convex decomposition, and it also has $\lceil n/2 \rceil - 1$ faces. However, not every polygon has a quadrangulation. Allowing Steiner points on the boundary of the polygon, Ramaswami, Ramos, and Toussaint [28] show that the minimum quadrangulation of every n-gon has at most $\lfloor 2n/3 \rfloor + O(1)$ faces in the worst case.

n	3	4	5	6	7	8	9	10	11	12	13	14	15
$\psi_c(n)$	1	1	2	2	2	2	2	3	3	3	3	3	3
$\psi_p(n)$	1	1	2	2	2	2	3	3	3	3	3..4	3..4	4
$\psi_d(n)$	1	2	2	3	4	4	5	6	6	7	8	8..9	8..9

Table 1: Bounds on the pseudo-convex cover number $\psi_c(n)$, partition number $\psi_p(n)$, and decomposition number $\psi_d(n)$ for small point sets.

2 Basic Combinatorial Properties

Our first (trivial) observation is that $\psi_d(n) \leq \kappa_d(n)$, $\psi_p(n) \leq \kappa_p(n)$, and $\psi_c(n) \leq \kappa_c(n)$. It is well known that $\kappa_c(n) \leq \kappa_p(n) \leq \kappa_d(n)$. For pseudo-convex faces we trivially have $\psi_c(n) \leq \psi_p(n)$. $\psi_p(n) \leq \psi_d(n)$ follows from the bounds given in the previous section.

Next we observe that $\psi_d(n+1) \leq \psi_d(n)+1$, $\psi_p(n+1) \leq \psi_p(n)+1$, and $\psi_c(n+1) \leq \psi_c(n)+1$. This follows by induction when inserting the points sorted accordint to their x-coordinates. For covering and partitioning, the last inserted vertex is a singleton (hence convex); for decomposing, the difference between the convex hull of the first n pooints and all $n+1$ points is a pseudo-triangle, which a corner at the last inserted point.

The following lemma establishes an interesting connection between the convex partition number and the pseudo-convex decomposition number.

Lemma 1 *For every finite point set S, we have $\psi_d(S) \leq 3\kappa_p(S) - 2$ and thus $\psi_d(n) \leq 3\kappa_p(n) - 2$.*

Proof. Every pointed pseudo-triangulation of S is a pseudo-convex decomposition of S with $n-2$ faces. We construct a pseudo-convex decomposition as follows: Take the $\kappa_p(S)$ polygonal domains of a minimum convex partition of S (some of which may be singletons or line segments) and pseudo-triangulate their complement in a pointed way. By triangulating each convex face, we can transform this pseudo-convex decomposition into a pseudo-triangulation: this show that we use ewer than $n-2$ faces. For a convex face of size $k_i \geq 3$, this transformation creates $k_i - 3$ additional faces.

Since each point of S is the vertex of exactly one face of a convex partition, we have $\sum_{i=1}^{\kappa_p(S)} k_i = n$, and so we can reduce the number of faces by at least $\sum_{i=1}^{\kappa_p(S)} (k_i - 3) = n - 3\kappa_p(S)$. Therefore, a minimum convex partition of S directly yields a pseudo-convex decomposition of S with at most $(n-2) - (n - 3\kappa_p(S)) = 3\kappa_p(S) - 2$ faces. □

The pseudo-convex decomposition, partition, and covering numbers for a particular point set S are not necessarily monotone. Consider the examples in Figure 2. On the left, a set S with 9 points and $\psi_d(S) = 3$. Removing the bottom-most point of S results in a set S' with 8 points and $\psi_d(S') = 4$. On the right, a set S with 6 points and $\psi_c(S) = \psi_p(S) = 1$. Removing the top-most point of S results in a set S' with 5 points and $\psi_c(S') = \psi_p(S') = 2$.

Table 1 shows the exact values of $\psi_c(n)$, $\psi_p(n)$, and $\psi_d(n)$ for small sets of points.

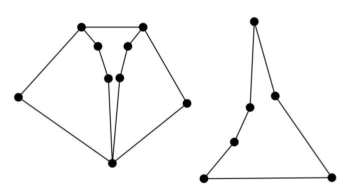


Figure 2: Sets with non-monotone behavior.

3 Pseudo-Convex Decompositions

We first give a formula for the number of faces in a pseudo-convex decomposition:

Lemma 2 *Let S be a set of n points in general position. Let P be a pseudo-convex decomposition of S, n_k the number of convex k-gons in P, and p the number of pointed vertices. Then the number of faces of P is*

$$|P| = 2n - p - 2 - \sum_{k=4}^{n} n_k(k-3).$$

Proof. We triangulate every convex k-gon in our decomposition (for $k \geq 4$) and thus obtain a pseudo-triangulation with $2n-p-2$ pseudo-triangles. Triangulating a convex k-gon introduces $k-3$ new faces. The proof follows. □

Corollary 3 *The number of faces in a pointed pseudo-convex decomposition is*

$$|P| = n - 2 - \sum_{k=4}^{n} n_k(k-3).$$

Although the pseudo-convex decomposition number for a particular point set S might not be monotone (recall Figure 2), $\psi_d(n)$ nevertheless increases monotonically with n.

Theorem 4 *The pseudo-convex decomposition number increases monotonically with the number of points.*

Proof. We want to show that $\psi_d(n) \leq \psi_d(n+1)$, which is equivalent to showing that for every point sets S with $|S| = n$, $\psi_d(S) \leq \psi_d(n+1)$ holds. Consider a set S of n points and let $q \in S$ be a point on the convex hull of S. We place a new vertex q^+ arbitrarily close to q to get the set $S^+ = S \cup q^+$ such that both q and q^+ lie on the convex hull of S^+. Note that $S^+ \setminus q$ has the same order type as S, that is, for any two points $p_1, p_2 \in S \setminus q$ the triples p_1, p_2, q and p_1, p_2, q^+ have the same orientation.

As S^+ has $n+1$ points, it has a pseudo-convex decomposition D^+ with at most $\psi_d(n+1)$ faces. Note that the face F in D^+ which contains the edge qq^+ has to be convex, otherwise P would be a pseudo-triangle and the points q and q^+ would lie on opposite sides of the line through one of the edges F. Now contract the edge qq^+ inot q. By this transformation the face F loses one edge, but all other faces of D^+ remain combinatorially unchanged, that is, either convex polygons or valid pseudo-triangles. Thus we obtain a pseudo-decomposition D of S which has either the same number of faces as D^+ or, in the case that F was a triangle, one fewer. Therefore $\psi_d(S) \leq \psi_d(S^+) \leq \psi_d(n+1)$. □

3.1 Two Geometric Ramsey-type Results

Let S be a planar point set in general position. We say that an *empty k-gon* is a simple polygon spanned by k points of S that contains no point of S in its interior.

Theorem 5 *Every set of* 8 *points in general position contains either an empty convex pentagon or two disjoint empty convex quadrilaterals.*

Theorem 6 *Every set of* 11 *points in general position contains either an empty convex hexagon or an empty convex pentagon and a disjoint empty convex quadrilateral.*

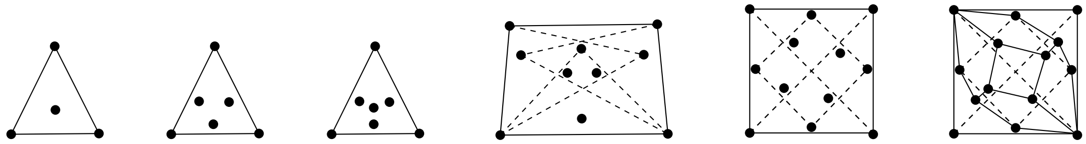
Both results were established with the help of the order type data base [2, 3]. In Subsection 3.5 we also provide a surprisingly intuitive geometric proof of Theorem 5 that requires only a moderate number of case distinctions.

3.2 Small Point Sets

In this section we give tight upper and lower bounds on $\psi_d(n)$ for sets of up to 13 points. Recall that $\psi_d(n+1) \leq \psi_d(n) + 1$ and (by Theorem 4) $\psi_d(n) \leq \psi_d(n+1)$. Obviously $\psi_d(3) = 1$. If four points do not lie in convex position (see Fig. 3(a)) then every decomposition needs at least two faces and hence $\psi_d(4) = 2$ and $\psi_d(5) \geq 2$. Every set of 5 points contains an empty convex quadrilateral [8]. Pseudo-triangulating in a pointed way around this quadrilateral yields $\psi_d(5) = 2$ by Corollary 3.

$\psi_d(5) = 2$ implies $\psi_d(6) \leq 3$. Figure 3(b) shows a configuration S of 6 points such that every pseudo-convex decomposition of S has at least 3 faces. S does not span any empty convex k-gon for $k > 4$. Every empty convex quadrilateral spanned by S necessarily uses all three inner points, so every partition of S can contain at most one convex quadrilateral. Then Corollary 3 implies $\psi_d(6) = 6 - 2 - (4-3) = 3$ for pointed pseudo-decompositions, which are optimal in this case.

$\psi_d(6) = 3$ implies $\psi_d(7) \leq 4$. Figure 3(c) shows a configuration S of 7 points such that every pseudo-convex decomposition of S has at least 4 faces. The argument is similar to the one for the example with 6 points. Again, S does not span any empty convex k-gon for $k > 4$. Every pointed decomposition contains at most one convex quadrilateral, because every convex quadrilateral contains the point in the center. Every additional quadrilateral forces at least one additional vertex to be non-pointed, so a non-pointed decomposition cannot contain fewer faces than a pointed one. Therefore, $\psi_d(7) = 7 - 2 - (4-3) = 4$.



(a) $n = 4$ (b) $n = 6$ (c) $n = 7$ (d) $n = 10$ (e) $n = 12$ (f) $n = 14$

Figure 3: (a)-(e) Lower bound examples, (f) every minimum decomposition is non-pointed.

$\psi_d(7) = 4$ implies $\psi_d(8) \geq 4$. Theorem 5 together with Corollary 3 implies $\psi_d(8) \leq 8 - 2 - 2 = 4$. We construct this decomposition by pseudo-triangulating in a pointed way the complement of the convex polygon(s) guaranteed by Theorem 5.

Every set of 10 points contains an empty pentagon [15] and so Corollary 3 implies $\psi_d(10) \leq 10 - 2 - (5-3) = 6$. Figure 3(d) (which is a close relative of a construction in [16]) shows a configuration S of 10 points such that every pseudo-convex decomposition of S has at least 6 faces. First note that S does not span any empty convex pentagon and a disjoint empty convex quadrilateral. Furthermore, every empty convex pentagon spanned by S necessarily contains the three points in the upper center, so every partition of S can contain at most one convex pentagon. If we start our decomposition with a pentagon, then we can not add a quadrilateral without creating at least one non-pointed vertex. Therefore, a non-pointed decomposition cannot have fewer faces than a the pointed one, which implies $\psi_d(10) = 10-2-(5-3) = 6$.

$\psi_d(10) = 6$ implies that $\psi_d(9) \geq 5$. Since every set of 9 points contains two disjoint empty convex quadrilaterals [16], we have (with Corollary 3) $\psi_d(9) \leq 9 - 2 - 2 \cdot (4-3) = 5$.

$\psi_d(10) = 6$ also implies $\psi_d(11) \geq 6$. Theorem 6 together with Corollary 3 yields $\psi_d(11) \leq 11-2-3 = 6$. We construct this decomposition by pseudo-triangulating in a pointed way around the convex polygon(s) guaranteed by Theorem 6.

$\psi_d(11) = 6$ implies $\psi_d(12) \leq 7$. Figure 3(e) shows a configuration S of 12 points such that every pseudo-convex decomposition of S has at least 7 faces. The largest empty convex set in this configuration is a hexagon. Every empty convex pentagon or hexagon contains at least three of the four inner points and thus separates the other points, so that no disjoint convex quadrilateral can be found. The coordinates of this point set are: $(0,0)$, $(0,20)$, $(20,20)$, $(20,0)$, $(1,10)$, $(10,19)$, $(19,10)$, $(10,1)$, $(5,7)$, $(7,15)$, $(15,13)$, $(13,5)$.

$\psi_d(12) = 7$ implies $\psi_d(13) \leq 8$. The point set with the following coordinates requires 8 faces for every pseudo-convex decomposition: $(65535, 65535)$, $(0,0)$, $(29293, 36890)$, $(15166, 26472)$, $(27461, 37283)$, $(32929, 42217)$, $(29439, 42711)$, $(27746, 42587)$, $(27491, 42925)$, $(32135, 45720)$, $(29447, 45175)$, $(31736, 48764)$, $(19257, 42830)$. This lower bound example was found with the help of the order type database [3].

Non-pointed Decompositions. All upper bounds on $\psi_d(n)$ for $n \leq 13$ can be achieved with pointed decompositions as described in the preceding paragraphs. Also the general upper bound can be realized with a pointed decomposition, as we will see in the next subsection. However, for $n \geq 10$, there are point sets such that an optimal (minimal) decomposition is always non-pointed. See, for example, Figure 3(f) which shows a configuration of 14 points such that every minimal pseudo-convex decomposition is non-pointed. The coordinates for this point set are the same as the ones for Figure 3(e) with the addition of $(4,5)$ and $(16,15)$.

3.3 Upper Bound

Our upper bound construction is based on exact pseudo-convex decomposition numbers for small point sets. Assume that we are given a set S with n points and that we know the value of $\psi_d(k)$ for some $k < n$. We choose a point p on the convex hull of S. Now we partition the plane by half-lines emanating from p into $\lceil (n-1)/(k-1) \rceil$ wedges such that every wedge contains at most $k-1$ points of $S \setminus \{p\}$. Let a *petal* be the convex hull of points in a wedge together with p. We have a total of $\lceil (n-1)/(k-1) \rceil$ petals, each of which can be decomposed into at most $\psi_d(k)$ faces. Two adjacent petals can be combined with a pseudo-triangle into one larger convex set. We combine inductively adjacent

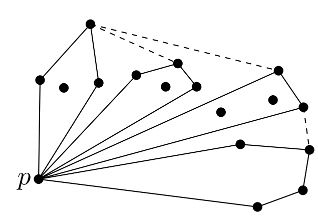


Figure 4: Petals of size 5.

convex sets (all including p) until we obtain the convex hull of S. We have proved an upper bound of

$$\psi_d(n) \leq \left\lceil \frac{n-1}{k-1} \right\rceil \psi_d(k) + \left\lceil \frac{n-1}{k-1} \right\rceil - 1 \leq \frac{\psi_d(k)+1}{k-1} n \ . \qquad (1)$$

The best currently known upper bound can be achieved by evaluating Inequality (1) for $k = 11$ and $\psi_d(11) = 6$. We obtain

$$\psi_d(n) \leq \frac{\psi_d(11)+1}{11-1} n = \frac{6+1}{10} n = \frac{7n}{10} \ .$$

Furthermore, the left inequality of (1) implies $\psi_d(15) \leq 9$ for $k = 8$.

3.4 Lower Bound

We present a lower bound construction of $5k$ points for every odd $k \geq 3$ such that any pseudo-convex decomposition consists of at least $3k - 1$ faces (see Fig. 5). This implies

$$\psi_d(n) \geq \frac{3n}{5} - 1.$$

Lemma 7 *For every odd k, there are $5k$ points in the plane such that every pseudo-convex decomposition consists of at least $3k - O(1)$ faces.*

Description of our construction. For every odd $k \in \mathbb{N}$, we construct a set of $5k$ points $P_k = \{a_i, b_i, c_i, d_i, e_i, : i = 1, 2, \dots, k\}$. The polygons $A = a_1 a_2 \dots a_k$ and $C = c_1 c_2 \dots c_k$ form two centrally symmetric regular k-gons such that $A \subset C$. Let o denote their center of symmetry. For every $i = 1, 2 \dots, k$, the quadrilateral $Q_i = a_i b_i c_i d_i$ is a rhombus, where the diagonal $a_i c_i$ is much longer than $c_i d_i$. Point e_i lies near the center of the rhombus $a_i b_i c_i d_i$ in the interior of the triangle $a_i b_i d_i \cap a_i c_i d_i$. The configurations $\{a_i, b_i, c_i, d_i, e_i\}$, $i = 1, 2, \dots, k$, are congruent. See Figure 5 for an example with $k = 5$. The ratio of the diameter of the polygons A and C are so close to 1 that any rhombus Q_i can be separated from the other rhombi by a straight line. Furthermore, we choose the ratio of the two diagonals of Q_i such that (1) every line passing through a_i (resp., c_i) and another point of $\{a_i, b_i, c_i, d_i, e_i\}$ intersects the line segment $d_j b_{j+1}$ for $j = i + \frac{k-1}{2} \mod k$. The point e_i is so close to the midpoint of $b_i d_i$ that (1) every line spanned by $\{b_i, d_i, e_i\}$ intersects the segments $c_{i-1} c_i$ and $c_i c_{i+1}$; (2) for every vertex $x \in S \setminus \{b_i, d_i, e_i\}$, the half-line $x e_i$ lies in the angular domain $\angle b_i x d_i < \pi$.

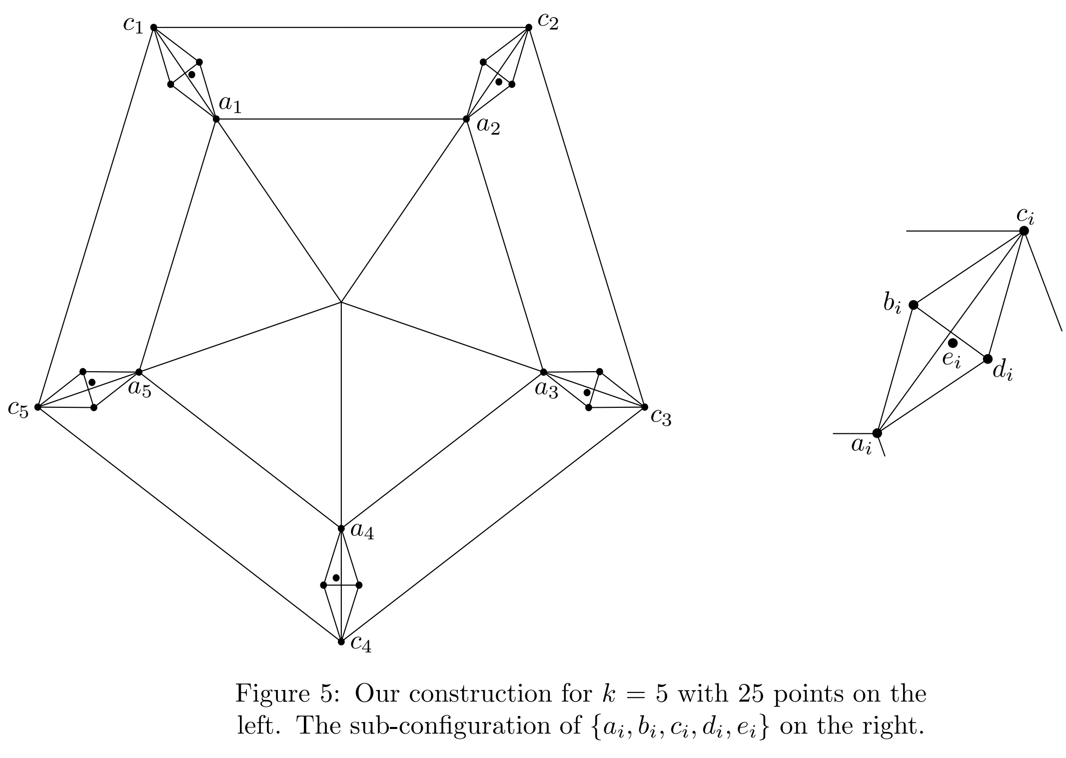


Figure 5: Our construction for $k = 5$ with 25 points on the left. The sub-configuration of $\{a_i, b_i, c_i, d_i, e_i\}$ on the right.

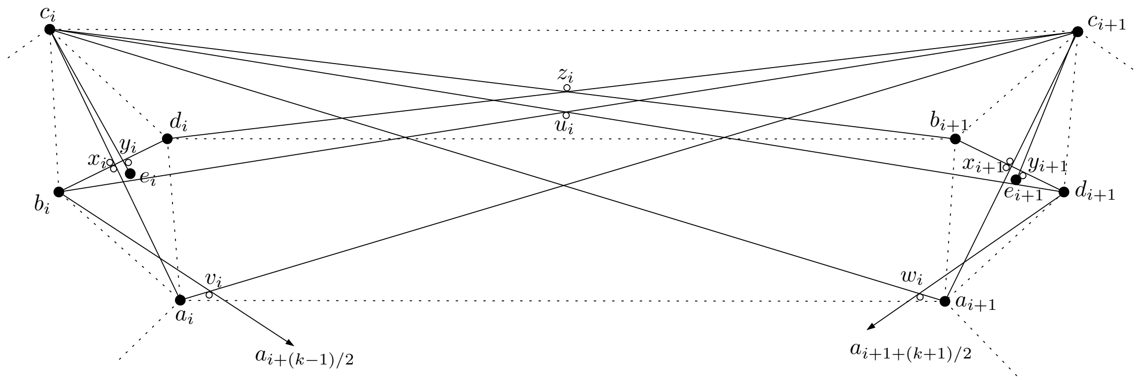


Figure 6: The location of the reference points for sub-configurations Q_i and Q_{i+1}.

Reference points. For a point set P_k and a pseudo-convex decomposition D, we choose $6k$ reference points and show that every face of D (with at most one exception) can contain at most two reference points. This proves that the number of faces is at least $3k - 1$.

Let $\varepsilon > 0$ be a sufficiently small real number. Each reference point lies in the ε-neighborhood of the intersection point of two lines determined by P_k, in a triangle incident to the intersection point. The locations of the six types of reference points are given in Table 2 below.

Reference point	in the ε-neighborhood of	in the triangle
x_i	$b_i d_i \cap a_i c_i$	$\Delta(a_i, b_i, b_i d_i \cap a_i c_i)$ or $\Delta(b_i, c_i, b_i d_i \cap a_i c_i)$
y_i	$b_i d_i \cap c_i e_i$	$\Delta(d_i, e_i, b_i d_i \cap c_i e_i)$
z_i	$c_i b_{i+1} \cap d_i c_{i+1}$	$\Delta(c_i, c_{i+1}, c_i b_{i+1} \cap d_i c_{i+1})$
u_i	$c_i d_{i+1} \cap b_i c_{i+1}$	$\Delta(b_i, d_{i+1}, c_i d_{i+1} \cap b_i c_{i+1})$
v_i	$a_i c_{i+1} \cap b_i a_{i+(k-1)/2}$	$\Delta(a_i, a_{i+(k-1)/2}, a_i c_{i+1} \cap b_i a_{i+(k-1)/2})$
w_i	$c_i a_{i+1} \cap d_{i+1} a_{i+1+(k+1)/2}$	$\Delta(a_{i+1}, a_{i+1+(k+1)/2}, c_i c_{i+1} \cap b_{i+1} a_{i+1+(k+1)/2})$

Table 2: The locations of the six types of reference points for $i = 1, 2, \ldots, k$ (addition is mod k).

The location of the reference points, however, depends on the pseudo-convex decomposition of P_k. The reference point x_i lies in the triangle $\Delta(b_i, c_i, b_i d_i \cap a_i c_i)$, if the decomposition contains the pseudo-triangle $b_i d_i c_i a_{i+1} e_i$ or $b_i c_i a_{i-1} e_i d_i$; otherwise x_i is in $\Delta(a_i, b_i, b_i d_i \cap a_i c_i)$. Also, if a pseudo-trianlge contains the vertices $a_i d_i c_i c_{i+1}$ in this order and another vertex, then we move the reference point v_i to w_i (which then has weight 2). Similarly, if a pseudo-trianlge contains the vertices $c_i c_{i+1} b_{i+1} a_{i+1}$ in this order and another vertex, then we move w_i to v_i.

Most faces can contain at most two reference points. No face contains more than four reference points. A face may contain three or four reference points if and only if it also contains the symmetry center o of the construction (Fig. 7(f)).

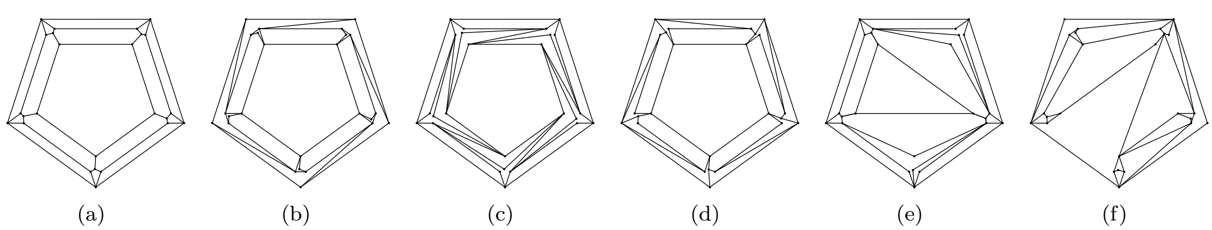


Figure 7: Five tilings of the convex hull of P_5 with 16 convex or pseudo-triangle faces (a–e), and one with 17 faces (f).

3.5 Proof of Theorem 5

Let S be a set of n points in general position in the plane. Recall that an empty k-gon is a polygon spanned by k points in S whose interior does not contain any points of S. Let $H \subset S$ be the set of points on the convex hull of S. We call the points of H the *outer points* and the points of $I = S \setminus H$ the *inner points* of S. In this section we prove Theorem 5, which we restate here for completeness.

Theorem 5 *Every set of* 8 *points in general position contains either an empty convex pentagon or two disjoint empty convex quadrilaterals.*

The proof of Theorem 5 consists of a case distinction based on the number of outer points.

Lemma 8 *If* $|H| \geq 6$ *then* S *contains an empty convex pentagon.*

Proof.

$|H| = 8$**:** S contains an empty convex octagon.

$|H| = 7$**:** There is a unique point $x \in I$. Choose an arbitrary $p \in H$. There are at least three point of H on one side of the line through p and x. Together with these three points, p and x form an empty convex pentagon.

$|H| = 6$**:** I consists of exactly two points x and y. They span a line that has at least three points of H on one side. Together with these 3 points, x and y form a convex pentagon.

□

Lemma 8 implies Theorem 5 for $|H| \geq 6$. To prove Theorem 5 for $|H| \leq 5$ we first collect several useful observations.

3.5.1 Observations and Definitions

We denote the convex hull of the inner points I of S by $P = \mathcal{CH}(I)$. Let H' be the vertex set of P. Two adjacent vertices q, r of P form a *face* $f = \{q, r\}$ of P. We say that a point $p \in H$ *sees* a face f of P or that f *is visible to* p if p and P are on different sides of the line l_f spanned by the vertices of f. If p sees f then we call the pair (p, f) a *visibility pair*. Let $V(p)$ be the set of faces visible to p, and let VP be the total number of visibility pairs, that is, $VP = \sum_{p \in H} |V(p)|$.

Lemma 9

(a) *For every* $p \in H$*, we have* $1 \leq |V(p)| \leq |H'| - 1$.

(b) *Every face* f *of* P *is visible to at least one vertex* $p \in H$.

Proof. (a) Consider a point $p \in H$ and draw two angent from p to P. By the general position assumption, there are two well-defined points $q_1, q_2 \in H'$, $q_1 \neq q_2$, incident to these two tangents. If $|I| \geq 3$, then the convex hull $P = \mathcal{CH}(I)$ is a closed polygonal curve; and the points q_1 and q_2 split P into two non-empty open polygonal curves. Point p sees every face of P along the polygonal curve closer to p, and it does not see any faces along the other polygonal curve.

(b) Consider a face f of P, and let $h(f)$ be the open halfplane bounded by the line through f that does not contain P. There must be an outer point in $h(f)$, otherwise the endpoints of f would be outer points. By definition, every point $p \in H \cap h(f)$ sees f. □

Observation 1 *The set* $\{p \in H \mid p \text{ sees } f\} \cup f$ *forms an empty convex polygon for every face* f *of* P.

Observation 1 immediately implies:

Observation 2 *If there are three or more vertices of* H *that see the same face* f *of* P*, then* S *contains an empty convex pentagon.*

Observation 2 and the pigeonhole principle imply:

Observation 3 *If* $VP > 2 \cdot |H'|$*, then* S *contains an empty convex pentagon.*

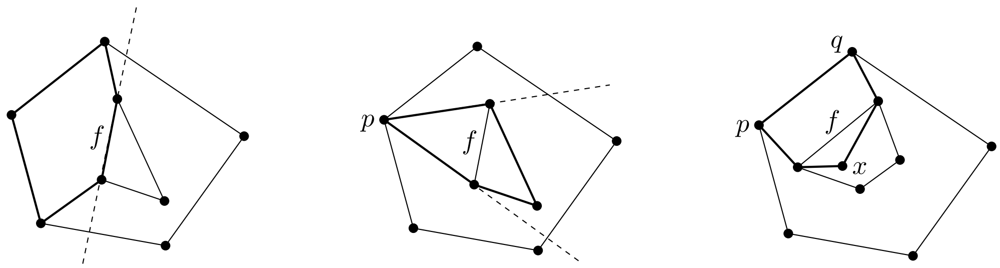


Figure 8: Observation 2 (left), Observation 4 (middle), and Observation 6 (right).

Next, we collect some properties of the outer vertices that see only one face of the inner polygon, that is, for every $p \in H$ such that $|V(p)| = 1$.

Observation 4 *$\{p \in H \mid V(p) = \{f\}\} \cup H'$ forms a convex polygon for every face f of P.*

Observation 4 implies:

Observation 5 *If $I \setminus H' = \emptyset$, then $\{p \in H \mid V(p) = \{f\}\} \cup H'$ forms an empty convex polygon for every face f of P.*

Observation 6 *If $I \setminus H' \neq \emptyset$, then there is a vertex $x \in I \setminus H'$ such that $\{p \in H \mid V(p) = \{f\}\} \cup f \cup \{x\}$ forms an empty convex polygon for every face f of P.*

If $|H'| \geq 3$ then Observation 5 and Observation 6 jointly imply:

Observation 7 *If $|H'| \geq 3$ and if there are two points p and q in H such that $V(p) = V(q) = \{f\}$ for some face f of P, then S contains an empty convex pentagon.*

And again by the pigeonhole principle:

Observation 8 *If $|H'| \geq 3$ and $|\{p \in H \mid |V(p)| = 1\}| > |H'|$, then S contains an empty convex pentagon.*

Now assume that there is an outer vertex $p \in H$ that sees two consecutive faces f_1 and f_2 of P; and let $p' = f_1 \cap f_2 \in H'$. The line $l_{p,p'}$ through p and p' partitions the plane into two halfplanes. One of them contains f_1, the other f_2. We call them $h_{f_1}(l_{p,p'})$ and $h_{f_2}(l_{p,p'})$, respectively.

Observation 9

(a) If $p, q \in H$, $f = \{p', q'\} \in V(p) \cap V(q)$, and there is an inner point in $h_f(l_{p,p'}) \cap h_f(l_{q,q'}) \cap P$, then p, q, p', q' and an inner point forms an empty convex pentagon.

(b) If $p \in H$, $f = \{p', q'\} \in V(p)$, there is an inner point contained in $h_f(l_{p,p'})$, and $q \in H$ such that $V(q) = \{f\}$, then S contains an empty convex pentagon.

Note that we assume that the cyclic orientation of the convex hull of both H and H' are the same (say, counter-clockwise), as indicated in Figure 9.

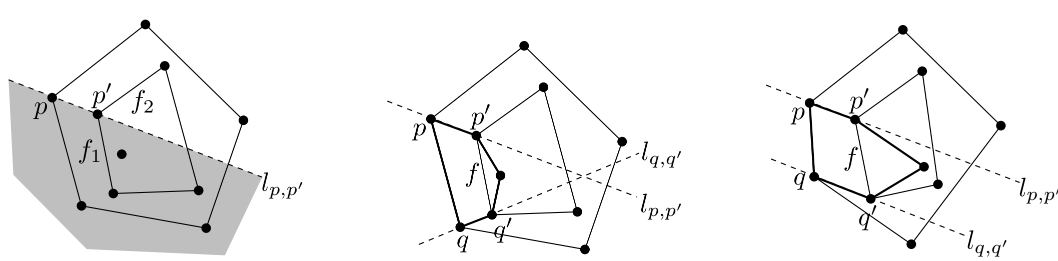


Figure 9: $h_{f_1}(l_{p,p'})$ (left), Observation 9(a) (middle), and Observation 9(b) (right).

3.5.2 Proof of Theorem 5 for $|H| \leq 5$

We now continue the case distinction for the proof of Theorem 5 based on the number of points in H and in H'.

$|H| = 5$

P necessarily is a triangle and every inner point belongs to H', that is, $|H'| = 3$. Lemma 9 implies that $|V(p)| \leq 2$ for every $p \in H$.

If there are at least two outer vertices that can see two faces of P, then $VP \geq 2 \cdot 2 + 3 = 7 > 6 = 2 \cdot |H'|$; and by Observation 3, S contains an empty convex pentagon.

Otherwise, there is at most one outer vertex that sees two faces, so four or more outer vertices can each see only one face of P. Since P has only three faces, $|\{p \mid |V(p)| = 1\}| > |\text{faces of } P|$ and by Observation 8, S contains an empty convex pentagon (see Fig. 10).

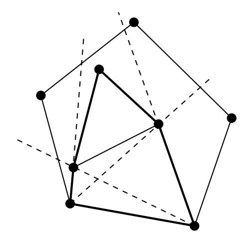


Figure 10: $|H| = 5$.

$|H| = 4$ and $|H'| = 4$

Since $|H'| = 4$, every inner point belongs to H', that is, $I \setminus H' = \emptyset$. Lemma 9 implies that $|V(p)| \leq 3$ for every $p \in H$.

If there is an outer vertex p such that $|V(p)| = 1$ then, by Observation 5, p forms an empty convex pentagon together with H'. Therefore we can assume that $|V(p)| \geq 2$ for every $p \in H$. If there is a $p \in H$ such that $|V(p)| = 3$, then $VP \geq 3 + 3 \cdot 2 = 9 > 8 = 2 \cdot |H'|$, so by Observation 3, S contains an empty convex pentagon. Thus we can assume that $|V(p)| = 2$ for every $p \in H$.

If there is a face f of P that is visible to more than 2 vertices then, by Observation 2, S contains an empty convex pentagon. So we can further assume that every face f of P is visible to exactly 2 outer vertices.

Let f be a face of P, and let p, q be the outer vertices that see f. By Observation 1, p, q and the vertices of f form an empty convex quadrilateral. The opposite face $f' = H' \setminus f$ is also visible to two vertices, but not to p or q, because each of them sees only two faces. Again by Observation 1, the remaining two outer vertices form a second quadrilateral together with the vertices of f' (see Fig. 11).

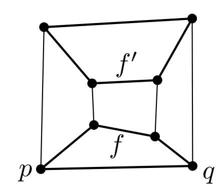


Figure 11: $|H| = 4$, $|H'| = 4$.

$|H| = 4$ and $|H'| = 3$

Since $|H'| = 3$, there is one inner point x that does not belong to H', that is, $I \setminus H' = \{x\}$. Lemma 9 implies that $|V(p)| \leq 2$ for every $p \in H$. We distinguish subcases according to the number of outer vertices that see two faces of P.

If there are more than two outer vertices that see two faces of P, then $VP \geq 3 \cdot 2 + 1 = 7 > 6 = 2 \cdot |H'|$ and by Observation 3, S contains an empty convex pentagon. If, on the other hand, $|V(p)| = 1$ for every $p \in H$, then, by the pigeonhole principle, one face f of P must be visible to two vertices, and by Observation 7, S contains an empty convex pentagon. We can, therefore, assume that either exactly one or exactly two outer vertices see two faces of P and that each of the remaining outer vertices see exactly one face of P. Furthermore, we can assume that if two outer vertices each see only one face of P then these two faces are different.

$|V(p)| = 2$ for exactly one outer vertex p**,** see Figure 12 (left).

Let $V(p) = \{f_1, f_2\}$. According to our assumptions, there are two outer vertices s and q such that $V(s) = \{f_1\}$ and $V(q) = \{f_2\}$. Since x must be contained in either $h_{f_1}(l_{p,p'})$ or $h_{f_2}(l_{p,p'})$ we can apply Observation 9(b) to either p, f_1, and s or p, f_2, and q and hence S contains an empty convex pentagon.

$|V(p)| = |V(q)| = 2$ for exactly two outer vertices p and q, see Figure 12 (right).

$|H'| = 3$ implies that $V(p) \cap V(q) \neq \emptyset$. If $V(p) = V(q)$ then necessarily at least one outer vertex $r \neq p, q$ sees a face from $V(p)$ and hence Observation 2 implies that S contains an empty convex pentagon.

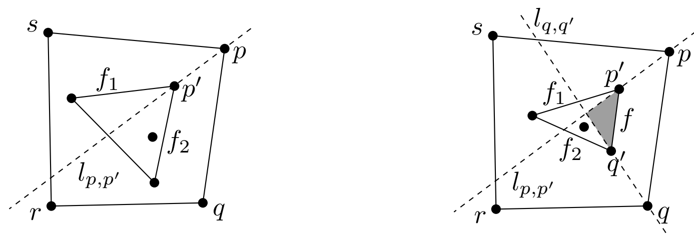


Figure 12: $|H| = 4$ and $|H'| = 3$, $h_f(l_{p,p'}) \cap h_f(l_{q,q'}) \cap P$ is shaded in the right figure.

Let therefore $V(p) = \{f, f_1\}$ and $V(q) = \{f, f_2\}$ with $f_1 \neq f_2$. If one of the remaining two outer vertices s and r sees f then Observation 2 again implies that S contains an empty convex pentagon. We can therefore assume that $V(s) = \{f_1\}$ and $V(r) = \{f_2\}$. If x is contained in $h_f(l_{p,p'}) \cap h_f(l_{q,q'}) \cap P$ then Observation 9(a) implies that S contains an empty convex pentagon. Otherwise x has to be contained in $h_{f_1}(l_{p,p'})$ or $h_{f_2}(l_{q,q'})$. We can apply Observation 9(b) to either p, f_1, and s or q, f_2, and r and hence S contains an empty convex pentagon.

$|H| = 3$

There are $|I| = 5$ inner points. If $|H'| = 5$ then the inner points form an empty convex pentagon. The remaining two cases are $|H'| = 4$ and $|H'| = 3$.

$|H| = 3$ and $|H'| = 4$

Since $|H'| = 4$ there is one inner point x that does not belong to H', that is, $I \setminus H' = \{x\}$. The diagonals of the inner quadrilateral P partition P into 4 regions. Each of them contains exactly one face of P. Let R_f be the region containing face f. Since we assume all points to be in general position x is contained in exactly one of these regions. Before we begin with a detailed case analysis we collect some additional observations.

Observation 10 *If $V(p) = \{f\}$ for $p \in H$ and $x \notin R_f$, then S contains an empty convex pentagon.*

Observation 1 and Observation 6 imply

Observation 11 *If $V(p) = \{f\}$ for $p \in H$ and the opposite face $f' = H' \setminus f \in V(q) \cap V(r)$ for $q \neq r \in H \setminus \{p\}$, then S contains two disjoint empty convex quadrilaterals.*

Observation 12 *If $V(p) = \{f_1, f_2\}$ for $p \in H$ with $f_1 \cap f_2 = p'$, $x \in h_{f_1}(l_{p,p'})$ and $f_3 = H' \setminus f_1 \in V(q) \cap V(r)$ for $q \neq r \in H \setminus \{p\}$, then S contains two empty convex quadrilaterals.*

Recall from Lemma 9 that every face f of P is visible to at least one vertex from H. Since $|H| = 3$ at least one of the outer vertices needs to see at least two faces. We distinguish the following subcases:

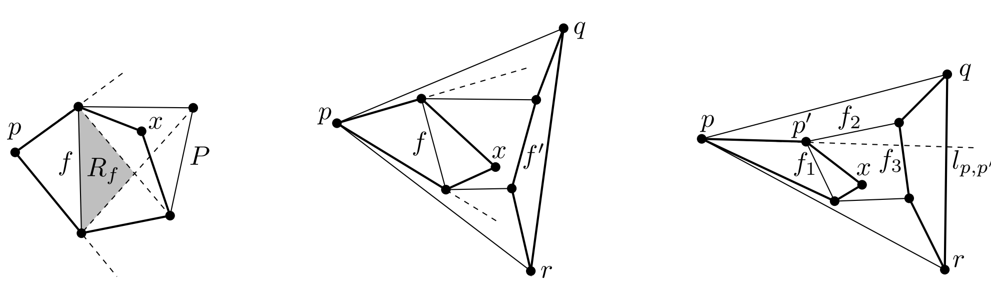
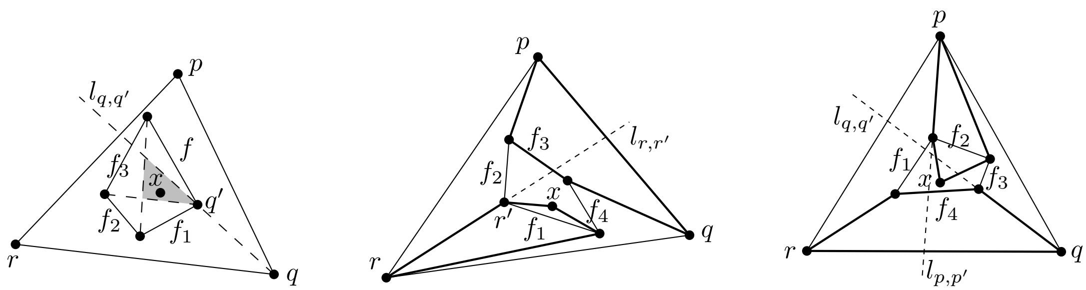


Figure 13: Observation 10 (left), Observation 11 (middle), and Observation 12 (right).



(a) $R_f \cap h_{f_1}(l_{q,q'})$ is shaded. (b) (c)

Figure 14: $|H| = 3$, $|H'| = 4$.

$|V(p)| = 1$ for at least one outer vertex p.

Let $V(p) = \{f\}$. If there is a second outer vertex q such that $V(q) = \{f\}$, then by Observation 7 S contains an empty convex pentagon. If there is a second outer vertex q such that $V(q) = \{f'\}$, where $f' \neq f$, Observation 10 can be applied to at least one of p and q, because x is contained in at most in one of R_f and $R_{f'}$ and hence S contains an empty convex pentagon.

We can therefore assume that the remaining outer vertices q and r both see at least two faces of P. If $\{f\} \in V(q) \cap V(r)$ then Observation 2 implies that S contains an empty convex pentagon. Let $f_2 = H' \setminus f$ denote the opposite face of f. If $\{f_2\} \in V(q) \cap V(r)$ then Observation 11 implies that S contains two disjoint empty convex quadrilaterals.

We can therefore assume w.l.o.g. that $\{f, f_1\} \subseteq V(q)$, $f_2 \notin V(q)$, $\{f_2, f_3\} \subseteq V(r)$, and $f \notin V(r)$ where f_1 and f_3 denote the two remaining, opposite faces of P (see Fig. 14(a)). Now consider the line $l_{q,q'}$ with $q' = f \cap f_1$. If $x \in h_f(l_{q,q'})$ then Observation 9(b) implies that S contains an empty convex pentagon. So we can assume that $x \in h_{f_1}(l_{q,q'})$. If $x \notin R_f$ then Observation 10 implies that S contains an empty convex pentagon. So we can also assume that $x \in R_f$. Since now $x \in h_{f_1}(l_{q,q'})$ and $x \in R_f$, that is, $R_f \cap h_{f_1}(l_{q,q'}) \neq \emptyset$, the two endpoints of f_2 must lie in $h_{f_1}(l_{q,q'})$ as well. They form an empty convex pentagon together with x and the points q, q'.

From now on we can assume that $|V(p)| \geq 2$ for every $p \in H$. Lemma 9 implies that $|V(p)| \leq 3$ for every $p \in H$. If $|V(p)| = 3$ for every $p \in H$ then $VP = 3 \cdot 3 = 9 > 8 = 2 \cdot |H'|$, so by Observation 3, S contains an empty convex pentagon. Thus we can assume that $|V(p) = 2|$ for at least one $p \in H$. By Observation 2 we can also assume that no face f of P is seen by all outer vertices.

$V(p) = V(q)$ for two outer vertices p and q.

Let r denote the third outer vertex. Since $|V(p)| \geq 2$ for every $p \in H$ and no face f of P is seen by all outer vertices we necessarily have $|V(p)| = |V(q)| = |V(r)| = 2$. Furthermore, r sees exactly the two faces of P that p and q do not see. Let $V(r) = \{f_1, f_2\}$ with $f_1 \cap f_2 = r'$ and $V(p) = V(q) = \{f_3, f_4\}$ (see Fig. 14(b)). Now consider the line $l_{r,r'}$. x must lie either in $h_{f_1}(l_{r,r'})$ or in $h_{f_2}(l_{r,r'})$. Since both $H' \setminus f_1$ and $H' \setminus f_2$ are contained in $V(p) \cap V(q)$ Observation 12 implies in either case that S contains two empty convex quadrilaterals.

$V(p) \cap V(q) = \emptyset$ for two outer vertices p and q.

Let r denote the third outer vertex. If either $V(p) = V(r)$ or $V(q) = V(r)$ then the previous case applies. So we can assume that $V(p) \neq V(r) \neq V(q)$. Necessarily $|V(p)| = |V(q)| = 2$ and $|V(r)| \in \{2, 3\}$. Let $V(p) = \{f_1, f_2\}$ with $p' = f_1 \cap f_2$ and $V(q) = \{f_3, f_4\}$ with $q' = f_3 \cap f_4$. There must be two faces $f \in V(r) \cap V(p)$ and $f' \in V(r) \cap V(q)$ with $f' \neq f$. W.l.o.g. $\{f_1, f_4\} \subseteq V(r)$ with $r' = f_1 \cap f_4$ (see Fig. 14(c)).

Since we assume all points to be in general position we know that x must lie either in $h_{f_1}(l_{r,r'})$ or in $h_{f_4}(l_{r,r'})$. If $x \in h_{f_1}(l_{p,p'}) \cap h_{f_4}(l_{q,q'})$ then Observation 9(a) applies to either p and r or q and r and implies that S contains an empty convex pentagon.

Let us assume that $x \notin h_{f_1}(l_{p,p'})$ which is equivalent to $x \in h_{f_2}(l_{p,p'})$. Since $f_4 = H' \setminus f_2$ is contained in $V(p) \cap V(r)$ Observation 12 implies that S contains two empty convex quadrilaterals.

Symmetrically, if $x \notin h_{f_4}(l_{q,q'})$ then necessarily $x \in h_{f_3}(l_{q,q'})$. Since $f_1 = H' \setminus f_3$ is contained in $V(q) \cap V(r)$ Observation 12 again implies that S contains two empty convex quadrilaterals.

$V(p) \neq V(q)$ and $V(p) \cap V(q) \neq \emptyset$ for any two outer vertices p and q.

Let p, q, and r denote the outer vertices. The condition above implies that one of them has to see 3 faces. W.l.o.g. let us assume that $|V(p)| = 3$. We also know that one of them sees only two faces. Again w.l.o.g. let us assume that $|V(q)| = 2$. $V(q)$ can not be a subset of $V(p)$ since that would imply that either one face of P is seen by all outer vertices or that $V(q)$ and $V(r)$ are disjoint. So let $V(p) = \{f_1, f_2, f_3\}$ and $V(q) = \{f_3, f_4\}$ with $q' = f_3 \cap f_4$. Necessarily $\{f_4, f_1\} \in V(r)$ and $f_3 \notin V(r)$.

If $f_2 \in V(r)$ then consider the line $l_{q,q'}$. x must lie either in $h_{f_3}(l_{q,q'})$ or in $h_{f_4}(l_{q,q'})$. Since both $f_1 = H' \setminus f_3$ and $f_2 = H' \setminus f_4$ are contained in $V(p) \cap V(r)$ Observation 12 implies in either case that S contains two empty convex quadrilaterals.

If $f_2 \notin V(r)$ then $V(r) = \{f_4, f_1\}$ with $f_4 \cap f_1 = r'$ (see Fig 15). If $x \in h_{f_4}(l_{q,q'}) \cap h_{f_4}(l_{r,r'})$ then Observation 9(a) implies that S contains an empty convex pentagon. Let us assume that $x \notin h_{f_4}(l_{q,q'})$, that is, $x \in h_{f_3}(l_{q,q'})$. Since $f_1 = H' \setminus f_3$ is contained in $V(p) \cap V(r)$ Observation 12 implies that S contains two empty convex quadrilaterals. Symmetrically, if $x \notin h_{f_4}(l_{r,r'})$, then necessarily $x \in h_{f_1}(l_{r,r'})$. Since $f_3 = H' \setminus f_1$ is contained in $V(p) \cap V(q)$ Observation 12 again implies that S contains two empty convex quadrilaterals.

$|H| = 3$ and $|H'| = 3$

Since $|H'| = 3$ there are two inner points x and y that do not belong to H', that is, $I \setminus H' = \{x, y\}$. These two inner points x and y span a line $l_{x,y}$. We say that $l_{x,y}$ *intersects* a face f, if the two vertices of f are on different sides of $l_{x,y}$. Since we assume all points to be in general position $l_{x,y}$ intersects exactly two of the three faces of P. Before we begin with a detailed case analysis we collect some additional observations.

Observation 13

(a) *If $V(p) = \{f\}$ and $l_{x,y}$ does not intersect f, then p, x, y and the two vertices of f form an empty convex pentagon.*

(b) *If $V(p) = \{f, f'\}$, $x, y \in h_f(l_{p,p'})$ and $l_{x,y}$ does not intersect f, then p, x, y and the two vertices of f form an empty convex pentagon.*

Lemma 9 implies that $|V(p)| \leq 2$ for every $p \in H$. We distinguish the subcases by the number of outer vertices that see two faces of P.

$|V(p)| = 1$ for every $p \in H$.

Every face f of P is seen by exactly one outer vertex. Since $l_{x,y}$ does intersect only two of the three faces of P Observation 13(a) implies that S contains an empty convex pentagon.

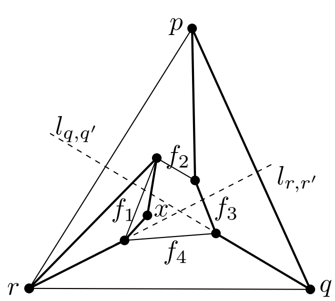


Figure 15: $|H| = 3$, $|H'| = 4$.

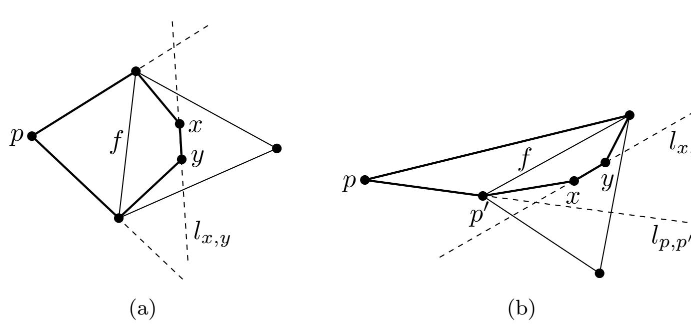


Figure 16: Observation 13.

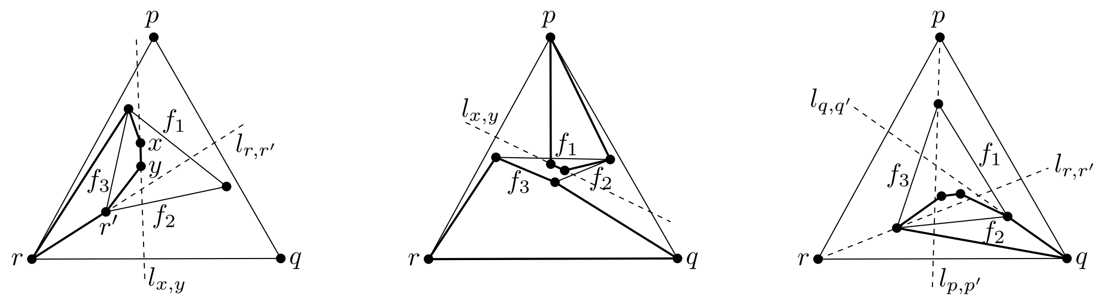


Figure 17: $|H| = 3$ and $|H'| = 3$.

$|V(p)| = |V(q)| = 1$ for exactly two outer vertices p and q, see Figure 17 (left).

If $V(p) = V(q)$, then Observation 7 implies that S contains an empty convex pentagon. We can therefore assume that $V(p) \neq V(q)$. Let r denote the remaining outer vertex, let $V(p) = \{f_1\}$, and let $V(q) = \{f_2\}$. W.l.o.g. we assume that $V(r) = \{f_2, f_3\}$ with $r' = f_2 \cap f_3$.

If $l_{x,y}$ does not intersect either f_1 or f_2 then Observation 13(a) implies that S contains an empty convex pentagon. Hence we can assume that $l_{x,y}$ does intersect both f_1 and f_2 and therefore does not intersect f_3. If either $x \in h_{f_2}(l_{r,r'})$ or $y \in h_{f_2}(l_{r,r'})$ then Observation 9(b) implies that S contains an empty convex pentagon. So we can assume that $x, y \in h_{f_3}(l_{r,r'})$ and then Observation 13(b) implies that S contains an empty convex pentagon.

$|V(p)| = 1$ for exactly one outer vertex p, Figure 17 (middle).

Let $V(p) = \{f_1\}$ and denote by q and r the remaining outer vertices. If $l_{x,y}$ does not intersect f_1 then Observation 13(a) implies that S contains an empty convex pentagon. Hence we can assume that $l_{x,y}$ does intersect f_1.

If $V(q) = V(r)$ then necessarily $V(q) = V(r) = \{f_2, f_3\}$. We can assume w.l.o.g. that $l_{x,y}$ intersects f_2. Let $p' = f_1 \cap f_2$. x, y, p and p' form an convex quadrilateral. Since $H' \setminus p' = f_3 \in V(q) \cap V(r)$ Observation 1 implies that the remaining four vertices from a convex quadrilateral as well.

If $V(q) \neq V(r)$ then we can assume that $V(q) = \{f_1, f_2\}$ with $q' = f_1 \cap f_2$ and $V(r) = \{f_2, f_3\}$ with $r' = f_2 \cap f_3$. If either x or y are contained in $h_{f_1}(l_{q,q'})$ then Observation 9(b) implies that S contains an empty convex pentagon. So we can assume that both x and y are contained in $h_{f_2}(l_{q,q'})$. Now if either x or y are contained in $h_{f_2}(l_{r,r'})$ then Observation 9(a) implies that S contains an empty convex pentagon. Thus we have $x, y \in h_{f_2}(l_{q,q'}) \cap h_{f_3}(l_{r,r'})$. Since $l_{x,y}$ intersects f_1 it can intersect only one of f_2 and f_3 and hence Observation 13(b) implies that S contains an empty convex pentagon.

$|V(p)| = 2$ for every $p \in H$, Figure 17 (right).

Let us denote the three outer vertices with p, q, and r. W.l.o.g. let $V(p) = \{f_3, f_1\}$ with $p' = f_3 \cap f_1$, $V(q) = \{f_1, f_2\}$ with $q' = f_1 \cap f_2$, and $V(r) = \{f_2, f_3\}$ with $r' = f_2 \cap f_3$. If either x or y is contained in either of $h_{f_1}(l_{p,p'}) \cap h_{f_1}(l_{q,q'})$, $h_{f_2}(l_{q,q'}) \cap h_{f_2}(l_{r,r'})$, or $h_{f_3}(l_{r,r'}) \cap h_{f_3}(l_{p,p'})$ then Observation 9(a) implies that S contains an empty convex pentagon. We therefore assume that $x, y \in h_{f_1}(l_{p,p'}) \cap h_{f_2}(l_{q,q'}) \cap h_{f_3}(l_{r,r'})$ or $x, y \in h_{f_3}(l_{p,p'}) \cap h_{f_1}(l_{q,q'}) \cap h_{f_2}(l_{r,r'})$. In either case, since $l_{x,y}$ intersects only two out of f_1, f_2, and f_3, Observation 13(b) implies that S contains an empty convex pentagon.

□

4 Pseudo-Convex Partitions

An upper bound of $\psi_p(n) \leq \lceil n/4 \rceil$ can be easily established: Partition the point set by vertical lines into groups of size four, allowing at most one group to be smaller if n is not a multiple of four—every four points forms either a convex quadrilateral or a pseudo-triangle. Any better bound than $\lceil n/4 \rceil$ for some $n \in \mathbb{N}$ may improve this upper bound. For example, we do not know the exact value of $\psi_p(13)$, we only know that $\psi_p(13) \in \{3, 4\}$ (c.f., Table 1). $\psi_p(13) = 3$ would imply $\psi_p(n) \leq \lceil 3n/13 \rceil$ by partitioning the point set into groups of size 13 and partitioning the groups independently.

4.1 Lower Bound

Lemma 10 $\psi_p(n) \geq \lfloor \frac{3n}{16} \rfloor$.

Proof. We consider a set S of $n = 4k$ points illustrated in Figure 18. S consists of k groups of 4 points, a_i, b_i, c_i, and d_i. First we show that if c_i is a reflex vertex of a pseudo-triangle P, then a_i and b_i must be the corners of P: This is the case since c_i lies in the convex hull of the corners of P, and there is a halfplane for a_i (b_i) whose boundary line passes through c_i and whose intersection with P is a_i (b_i).

Let $W \subset S$ denote a subset of $3k$ points $\{a_i, b_i, c_i : i = 1, 2, \ldots, k\}$. Consider a polygon P from a pseudo-convex partition of S. We show next that P is incident to at most 4 points of W. This implies immediately that every pseudo-convex partition of S consists of at least $3k/4 = 3n/16$ polygons. Suppose, by contradiction, that P is incident to more than 4 points of W.

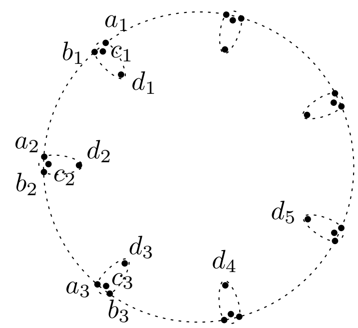


Figure 18: $k = 7$.

First suppose that P is convex, that is, P contains a convex pentagon Q with all vertices in W. Since each group contains only three points of W, Q must have corners in at least two groups. Q can contain at most two points from each group, because the triangle $a_ib_ic_i$ cannot be completed to a convex pentagon in S. Therefore, Q must have corners in at least three groups, and it contains a triangle T with corners of W from three different groups. There is a group $i \in \{1, 2, \ldots, k\}$ such that T has a corner in group i and the other two corners of T are in a group $j \in [i+1, i+\lfloor k/2 \rfloor \mod k]$ and in a group $j' \in [i+\lceil k/2 \rceil \mod k, i+k-1 \mod k]$. It follows that T contains d_i in its interior, a contradiction.

If P is a pseudo-triangle with at least five vertices in W, then it must have at least two reflex vertices in W. Since the convex hull vertices can only be corners of P, at least two reflex vertices are some c_i and c_j, $i \neq j$. We have seen that if P contains c_i and c_j, then it also contains a_i, b_i and a_j, b_j, and so it must have four corners: A contradiction. □

5 Pseudo-Convex Coverings for Small Point Sets

Lemma 11 *Every set S of 6 points, not all on a line, has a spanning pseudo-triangle.*

Lemma 12 *Every set S of $n \in \mathbb{N}$ points contains either a convex hexagon or a pseudo-triangle with 6 vertices if and only if $n \geq 12$.*

Proof. Let h be the size of the convex hull of S. We prove the statement by a case analysis over various values of h. If $h = 3$, then the theorem directly follows from Lemma 11. Note that additional inner points are allowed in Lemma 11, as we do not require the hexagon or empty pseudo-triangle to be empty. For $h = 4, 5$, decompose the convex hull of S into two (three, respectively) triangles. At least one triangle must contain 3 or more interior points. Again the our statement follows from Lemma 11 applied to this triangle. If $h \geq 6$, then S certainly contains a convex hexagon.

There exist precisely 9 (out of over 2.33 billion) realizable order types of 11 points which do not contain a convex hexagon nor a pseudo-triangle with 6 vertices[1]. Thus the bound is tight with respect to n. □

Lemma 13 $\psi_c(n) = 3$ *for* $n = 10, \ldots, 15$.

Proof. For $n = 10, \ldots, 14$, we obtain $\psi_c(n) \leq 3$ from the fact that every set of $n \geq 9$ points contains a convex pentagon together with $\psi_c(5) = 2, \ldots, \psi_c(9) = 2$. The matching lower bound $\psi_c(10) \geq 3$ (and thus $\psi_c(11) \geq 3, \ldots, \psi_c(14) \geq 3$) follows from a configuration of 10 points whose pseudo-convex cover consists of at least 3 polygons. We found this configuration with the help of the order type data base and to our surprise there is only one set (out of 14,309,547 order types) which has this property. Here are its coordinates: (0,43470), (20468,62019), (27350,61551), (32984,63477), (34692,42743), (50624,39069), (64372,33534), (15064,31131), (16660,25083), (19152,0) (see Fig. 19).

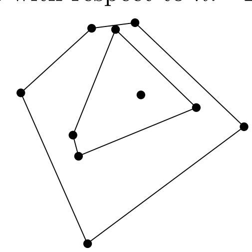


Figure 19: $n = 10$.

We obtain $\psi_c(15) = 3$ from Lemma 12 together with $\psi_c(9) = 2$. □

[1] Let us note here that the probability of winning the Jackpot of the Austrian lottery (6 out of 45) is about 30 times higher than the probability of finding such a set at random.

6 Pseudo-Convex Decompositions of the Interior of a Simple Polygon

Theorem 14 *Every simple polygon with $n \geq 3$ vertices has a decomposition into at most $\lceil \frac{n-2}{2} \rceil$ convex or pseudo-triangular faces, and this is the best possible bound.*

Proof. The lower bound is attained by the comb polygons (Fig. 20 (a)). We prove the upper bound by induction on $n \in \mathbb{N}$. The theorem is obvious for $n = 3, 4$. Consider a simple polygon P_n with $n \geq 5$ vertices. Triangulate P_n and let T_n denote the dual graph of the triangulation. Every node of T_n corresponds to a triangle, and every edge of T_n corresponds to a diagonal in the triangulation. T_n is a tree with maximal degree three and with $n-2$ nodes.

If n is odd then we delete a triangle t corresponding to a leaf node in T_n. By induction, $P_n - t$ can be decomposed into $\frac{n-3}{2}$ faces. Therefore P_n decomposes into $\frac{n-3}{2} + 1 = \lceil \frac{n-2}{2} \rceil$ faces.

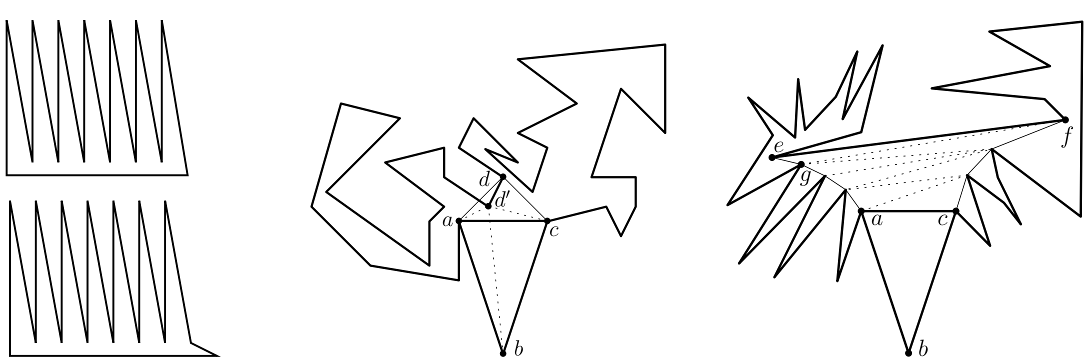
Assume that n is even, and so $\lceil \frac{n-2}{2} \rceil = \frac{n}{2} - 1$. The triangulation consists of an even number of triangles. If a diagonal decomposes P_n into two even polygons, then induction completes the proof. Hence we assume that every diagonal decomposes P_n into two odd polygons.

Let the triangle abc correspond to a leaf in T_n such that ac is a diagonal of P_n. We show that no diagonal of P_n is incident to b. Suppose, by contradiction, that bd is a diagonal of P_n (Fig. 20 (b)). Then $abcd$ must be a convex polygon. Let d' be the vertex of P_n in $acd \setminus \{a, c\}$ closest to the line ac. Note that bd' is a diagonal of P_n, and at least one of ad' and cd' is also a diagonal (since $n \geq 5$). If bd' decomposes P_n into odd polygons, then either ad' or cd' decomposes it into two (non-empty) even polygons. We conclude that b is not incident to any diagonal of P and so it sees the interior of a unique edge ef of P_n (Fig. 20 (c)).

Consider the pseudo-triangle $\mathrm{pt}(b, e, f)$ (three corners uniquely define a pseudo-triangle in a simple polygon). If $P_n = \mathrm{pt}(b, e, f)$, then P_n is a pseudo-triangle, and our proof is complete. Each of the components of $P_n - \mathrm{pt}(b, e, f)$ is an odd polygon. Every such component is adjacent to a unique edge of the geodesic $\mathrm{geo}(a, e)$ or $\mathrm{geo}(c, f)$. If $\mathrm{pt}(b, e, f)$ has k vertices, then it has $k-3$ edges along these geodesics (all edges except ab, bc, and ef). We show that there is one edge along the geodesics $\mathrm{geo}(a, e)$ and $\mathrm{geo}(c, f)$ that is not adjacent to any component of $P_n - \mathrm{pt}(b, e, f)$: Consider the dual graph of an arbitrary triangulation of $\mathrm{pt}(b, e, f)$. It is a tree where one leaf node corresponds to abc and another leaf corresponds to efg for some vertex g. Assume w.l.o.g. that eg is a side and fg is a diagonal in $\mathrm{pt}(b, e, f)$. If eg were adjacent to an odd component of $P_n - \mathrm{pt}(b, e, f)$, then fg would partition P_n into two even polygons. Therefore $\mathrm{pt}(b, e, f)$ with k vertices is adjacent to at most $k-4$ components of $P_n - \mathrm{pt}(b, e, f)$.

Let n_i denote the number of vertices of the components of $P_n - \mathrm{pt}(b, e, f)$ for $i = 1, 2, \ldots, k-4$. We have $k + \sum_{i=1}^{k-4}(n_i - 2) = n$. By induction, every odd component with n_i vertices can be decomposed into $(n_i - 1)/2$ faces. Together with $\mathrm{pt}(b, e, f)$, the polygon P_n can be decomposed into

$$1 + \sum_{i=1}^{k-4} \frac{n_i - 1}{2} \leq 1 + \frac{1}{2}\left(\sum_{i=1}^{k-4} n_i - 2\right) + \frac{k-4}{2} = \frac{n}{2} - 1$$



(a) The comb polygon for n odd and for n even.

(b) If b is incident to a diagonal bd, then there is a vertex d' such that bd' and at least one of ad' or cd' are also diagonals.

(c) If b sees the edge ef then we can form the pseudo-triangle $\mathrm{pt}(b, e, f)$.

Figure 20: Lower bound (a). An example 24-gon. (b)-(c).

faces, as required. □

7 Conclusions and Open Problems

We proposed pseudo-convex decompositions, partitions, and coverings. We established some of their basic properties and gave combinatorial bounds on their complexity. Our upper bounds depend on new Ramsey-type results concerning disjoint empty convex k-gons in the plane. We (obviously) would like to know what the exact bounds on $\psi_d(n)$ and $\psi_p(n)$ are and if the exact bound for $\psi_d(n)$ can be realized with a pointed decomposition. It would also be interesting to determine the complexity of computing a minimum pseudo-convex decomposition or covering for a given point set.

Acknowledgements. The first two authors thank Ferran Hurtado and Hannes Krasser for valuable discussions over the matter presented here.

References

[1] P. K. Agarwal, J. Basch, L. J. Guibas, J. Hershberger, and L. Zhang. Deformable free space tilings for kinetic collision detection. *International Journal of Robotics Research*, 21:179–197, 2002.

[2] O. Aichholzer, F. Aurenhammer, and H. Krasser. Enumerating order types for small point sets with applications. *Order*, 19:265–281, 2002.

[3] O. Aichholzer and H. Krasser. Abstract order type extensions and new results on the rectilinear crossing number. In *Proc. 21st Symposium on Computational Geometry*, pages 91–98, AMS Press, 2005.

[4] E. M. Arkin, S. P. Fekete, F. Hurtado, J. S. B. Mitchell, M. Noy, V. Sacristán, and S. Sethia. On the reflexivity of point sets. In *Discrete and Computational Geometry: The Goodman-Pollack Festschrift (B. Aronov et al., eds.)*, vol. 25 of Algorithms Combin., pages 139–156, Springer-Verlag, Berlin, 2003.

[5] B. Chazelle and D. Dobkin. Optimal convex decompositions. In *Computational Geometry* (G. T. Toussaint, ed.), pages 63–133, North-Holland, Amsterdam, 1985.

[6] B. Chazelle, H. Edelsbrunner, M. Grigni, L. J. Guibas, J. Hershberger, M. Sharir, and J. Snoeyink. Ray shooting in polygons using geodesic triangulations. *Algorithmica*, 12:54–68, 1994.

[7] M. de Berg, J. Matoušek, and O. Schwarzkopf. Piecewise linear paths among convex obstacles. *Discrete and Computational Geometry* 14(1):9–29, 1995.

[8] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935.

[9] P. Erdős and G. Szekeres. On some extremum problem in geometry. *Annales Univ. Sci. Budapest*, 3-4:53–62, 1960.

[10] T. Fevens, H, Meijer, and D. Rappaport. Minimum convex partition of a constrained point set. *Discrete Applied Mathematics*, 109:95–107, 2001.

[11] J. García-López and M. Nicolás. Planar point sets with large minimum convex partitions. In *Abstracts 22nd European Workshop on Computational Geometry*, pages 51–54, Delphi, 2006.

[12] S. Gerdjikov and A. Wolff. Pseudo-convex decomposition of simple polygons. In *Abstracts 22nd European Workshop on Computational Geometry*, pages 13–16, Delphi, 2006.

[13] J. Goodman and R. Pollack. Allowable sequences and order types in discrete and computational geometry. In *New Trends in Discrete and Computational Geometry*, pages 103–134, Springer-Verlag, New York, 1993.

[14] M. Goodrich and R. Tamassia. Dynamic ray shooting and shortest paths in planar subdivision via balanced geodesic triangulations. *Journal of Algorithms*, 23:51–73, 1997.

[15] H. Harborth. Konvexe Fünfecke in ebenen Punktmengen. *Elemente der Mathematik*, 33(5):116–118, 1978.

[16] K. Hosono and M. Urabe. On the number of disjoint convex quadrilaterals for a planar point set. *Computational Geometry: Theory and Applications*, 20:97–104, 2001.

[17] Gy. Károlyi, J. Pach and G. Tóth. Ramsey-type results for geometric graphs I. *Discrete and Computational Geometry*, 18:247–255, 1997.

[18] Gy. Károlyi, J. Pach, G. Tóth, and P. Valtr. Ramsey-type results for geometric graphs II. *Discrete and Computational Geometry*, 20:375–388, 1998.

[19] J. M. Keil. Decomposing a polygon into simpler components, *SIAM Journal on Computing*. 14:799-817, 1985.

[20] J. M. Keil and J. Snoeyink. On the time bound for convex decomposition of simple polygons. *International Journal of Computational Geometry and Applications* 12:181–192, 2002.

[21] D. Kirkpatrick and B. Speckmann. Kinetic maintenance of context-sensitive hierarchical representations for disjoint simple polygons. In *Proc. 18th Symposium on Computational Geometry*, pages 179–188, AMS Press, 2002.

[22] D. Larman, J. Matoušek, J. Pach, and J. Törőcsik, A Ramsey-type result for planar convex sets, *The Bulletin of the London Mathematical Society*, 26:132–136, (1994).

[23] J-M. Lien and N. M. Amato. Approximate convex decomposition. In *Proc. 20th Symposium on Computational Geometry*, pages 17-26, AMS Press, 2004.

[24] V. Neumann-Lara, E. Rivera-Campo, and J. Urrutia. A note on convex decompositions of a set of points in the plane. *Graphs and Combinatorics*, 20(2):223–231, 2004.

[25] J. O'Rourke. Visibility. In *Handbook of Discrete and Computational Geometry* (2nd ed.), pages 643–664, CRC Press, Boca Raton, FL, 1997.

[26] M. Pocchiola and G. Vegter. Topologically sweeping visibility complexes via pseudo-triangulations. *Discrete and Computational Geometry*, 16:419–453, 1996.

[27] M. Pocchiola and G. Vegter. Minimal tangent visibility graphs. *Computational Geometry: Theory and Applications*, 6:303–314, 1996.

[28] S. Ramaswami, P. A. Ramos, and G. T. Toussaint. Converting triangulations to quadrangulations. *Computational Geometry: Theory and Applications*, 9:257–276, 1998.

[29] A. Spillner. Optimal convex partitions of point sets with few inner points. In *Proc. 17th Canadian Conference on Computational Geometry*, pages 39–42, Windsor, ON, 2005.

[30] I. Streinu. A combinatorial approach to planar non-colliding robot arm motion planning. In *Proc. 41st Symposium on Foundations of Computer Science*, pages 443–453, IEEE Press, 2000.

[31] M. Urabe. On a partition into convex polygons. *Discrete Applied Mathematics*, 64:179–191, 1996.