# Incidences of not too degenerate hyperplanes

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**Abstract**

We present a multi-dimensional generalization of the Szemerédi-Trotter Theorem, and give a sharp bound on the number of incidences of points and *not-too-degenerate* hyperplanes in three- or higher-dimensional Euclidean spaces. We call a hyperplane *not-too-degenerate* if at most a constant portion of its incident points lie in a lower dimensional affine subspace.

## 1 Introduction

Settling a conjecture by Erdős, it was proved by Szemerédi and Trotter [15] that $n$ points and $\ell$ lines in the plane determine $O(n^{2/3}\ell^{2/3} + n + \ell)$ point-line incidences. This bound is tight apart from a constant factor (e.g., on a rectangular grid [10]). The Szemerédi-Trotter Theorem has innumerable applications in combinatorial and computational geometry. The methods developed for proving it (such as $\varepsilon$-cuttings [5] and crossing numbers [16], among others) had significant impact on many other combinatorial geometry problems. For the various proof techniques, extensions, and related problems, we refer the reader to an excellent recent survey by Pach and Sharir [14].

In the last two decades, tenacious research efforts uncovered several Szemerédi-Trotter type bounds on the number of point-curve incidences in the plane and point-surface incidences in higher-dimensional Euclidean spaces. Sharp bounds were obtained in only very few, specialized cases [14].

The focus of our paper is the number of point-hyperplane incidences in $\mathbb{R}^d$, $d \geq 3$. Note that $n$ points and $\ell$ hyperplanes in $\mathbb{R}^d$ may have $n\ell$ incidences if all $n$ points lie on a $(d-2)$-flat, and all $\ell$ hyperplanes contain this flat. Therefore, every non-trivial bound on point-hyperplane incidences in $\mathbb{R}^d$, $d \geq 3$, must make some reasonable restriction on the point-hyperplane configurations. It is a challenge to find conditions that still include most of the interesting point-hyperplane instances and allow for non-trivial, sharp, and provable bounds.

**Our contribution.** We propose a natural class of hyperplanes for which we can prove *sharp* bounds on the number of point-hyperplane incidences. In three-space, we call a plane *not-too-degenerate* if at most a constant portion of its incident points lie on a line. We show that for $n$ points and $\ell$ not-too-degenerate planes in $\mathbb{R}^3$, the number of incidences is $O(n^{3/4}\ell^{3/4} + n\sqrt{\ell} + \ell)$. This bound is tight apart from a constant factor.

Our result extends, in a weaker form, to higher dimensions. We call a hyperplane $\alpha$-degenerate if at most an $\alpha$ portion of its incident points lie in a $(d-2)$-dimensional flat. For every $d \geq 3$, there is a constant $\alpha_d \in (0,1)$ such that the number of incidences between $n$ points and $\ell$ $\alpha$-degenerate hyperplanes, $\alpha < \alpha_d$, in $\mathbb{R}^d$ is $O(n^{\frac{d}{d+1}}\ell^{\frac{d}{d+1}} + n\ell^{\frac{d-2}{d-1}} + \ell)$, and this bound is also best possible apart

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from a constant factor. As an easy corollary, we also derive an alternative proof for the known sharp bound $O(n^{d/3}\ell^{2/3}+n^{d-1})$ on the number of point-hyperplane incidences in $\mathbb{R}^d$ if every hyperplane is *spanned* by the point set (i.e., it contains $d$ affine independent points).

**Related previous results.** First Edelsbrunner, Guibas, and Sharir [7] studied the number of point-plane incidences in three-space. They obtained an (essentially) $O(n^{3/5}\ell^{4/5}+n+\ell\log n)$ bound assuming that there are no three collinear points. Braß and Knauer [4] considered the number of point-hyperplane incidences in $d$-dimensions and derived an upper bound of $O(n^{\frac{d}{d+1}}\ell^{\frac{d}{d+1}}\log(n\ell)+(n+\ell)\log(n+\ell))$ under the condition that the incidence graph does not contain a $K_{r,r}$ for some fixed $r\in\mathbb{N}$. These bounds are not known to be tight. Under the condition of [4], the best known lower bound is $\Omega(n^{7/10}\ell^{7/10})$ in three-space; and $\Omega((n\ell)^{1-\frac{2}{d+3}-\delta})$ and $\Omega((n\ell)^{1-2(d+1)/(d+2)^2-\delta})$, respectively, in odd and even dimensions in $\mathbb{R}^d$, $d\geq 4$, for any constant $\delta>0$.

One of the rare cases where a tight bound is known for point-hyperplane incidences in $\mathbb{R}^d$ is that all $\ell$ hyperplanes are spanned by the point set. Agarwal and Aronov [1] proved an $O(n^{d/3}\ell^{2/3}+n^{d-1})$ upper bound in $\mathbb{R}^d$, matching the lower bounds of Edelsbrunner and Haussler [6, 8]. A much more restrictive scenario was considered by Edelsbrunner and Sharir [9]: The number of point-hyperplane incidences in $\mathbb{R}^4$ with the condition that every point lies on the lower envelope of the hyperplanes is $O(n^{2/3}\ell^{2/3}+n+\ell)$, which is sharp, too.

We also mention the algorithmic sibling of the incidence problem, Hopcroft's problem, which prompted a strain of research in the computational geometry community (e.g., [11, 4]): Given $n$ points and $\ell$ hyperplanes in $\mathbb{R}^d$, decide if there is a point-hyperplane incidence (or in other variants, compute the number of incidences or report all point-line incidences).

**Organization.** In the next section, we present precise definitions for *not-too-degenerate* and *saturated* hyperplanes in $\mathbb{R}^d$, $d\geq 2$. We formulate our main result for saturated hyperplanes in Subsection 2.1. Then in Subsection 2.2, we break it down into two bounds for not-too-degenerate hyperplanes, one in $\mathbb{R}^3$ and another one in $\mathbb{R}^d$, $d\geq 4$. An elegant short proof for a bound involving hyperplanes spanned by the point set is presented in Subsection 2.3. We prove our three-dimensional upper and lower bounds in Section 3. Finally, we proceed with the proof of the matching upper and lower bounds on saturated hyperplanes in $\mathbb{R}^d$ in Section 4.

## 2 Formulation of Results

Given a point set $P\subset\mathbb{R}^d$ and a positive integer $k$, we say that a hyperplane $S$ is *$k$–rich* if $|S\cap P|\geq k$. Instead of the number of point-hyperplane incidences, we consider the number of $k$-rich hyperplanes for a set of $n$ points in $\mathbb{R}^d$. This allows us to talk about properties of a point set in $\mathbb{R}^d$, rather than a set of points *and* a set of hyperplanes. An *equivalent* formulation of the Szemerédi-Trotter Theorem [15], which we will use in our arguments, is the following.

**Theorem 2.1 (Szemerédi–Trotter [15])** *There is an absolute constant $C_0>0$ such that for any set of $n$ points in the plane, the number of $k$-rich lines, $k\geq 2$, is at most*

$$C_0\cdot\Big(\frac{n^2}{k^3}+\frac{n}{k}\Big).$$

*This bound is best possible apart from the constant factor.*

There is an easy correspondence between the two formulations. In general for $a, b \in \mathbb{N}$ in $\mathbb{R}^d$, an $O(n^a/k^b)$ bound on the number of $k$-rich hyperplanes, $k \geq k_0$ for an appropriate constant $k_0$, is equivalent to an $O(n^{a/b}\ell^{(b-1)/b})$ bound on the number of incidences between $n$ points and $\ell$ (at least) $k_0$-rich hyperplanes.

## 2.1 Saturated Hyperplanes

A set of $k$ points in general position in an $r$-dimensional Euclidean space spans $\binom{k}{r}$ distinct $(r-1)$-flats ($(r-1)$-dimensional affine subspaces). If the point set is *degenerate*, then they span one or no $(r-1)$-flats. In the following definition, we "measure" the degeneracy of an $r$-flat $F^{(r)}$ with respect to a point set $P$ in $\mathbb{R}^d$, $0 < r \leq d$, by the number of $(r-1)$-flats spanned by $F^{(r)} \cap P$.

**Definition 2.2** For integers $r, d \in \mathbb{N}$, $0 < r \leq d$, we are given a point set $P$ and an $r$-flat $F^{(r)}$ in $\mathbb{R}^d$. We say that $F^{(r)}$ is *$\gamma$-saturated*, $\gamma > 0$, if the point set $F^{(r)} \cap P$ spans at least $\gamma \cdot |F^{(r)} \cap P|^r$ distinct $(r-1)$-subflats of $F^{(r)}$.

Obviously, every $\gamma$-saturated hyperplane contains at least $d-1$ affine independent points of $P$. A $k$-rich hyperplane contains $d$ affine independent points (it is spanned by $P$) if $\gamma k^r > 1$, that is, if the hyperplane is at least $\gamma^{-1/r}$-rich. We are typically interested in the number of much richer hyperplanes. Our main result is a *tight* bound on the number of $k$-rich $\gamma$-saturated hyperplanes in any dimension $d \geq 2$.

**Theorem 2.3** *For any integer $d \geq 2$ and a real $\gamma > 0$, there is a constant $C_1(d, \gamma) > 0$ with the following property. For every set $P$ of $n$ points in $\mathbb{R}^d$, the number of $k$-rich $\gamma$-saturated hyperplanes is at most*

$$C_1(d,\gamma)\left(\frac{n^d}{k^{d+1}} + \frac{n^{d-1}}{k^{d-1}}\right).$$

*This bound is best possible apart from the constant factor $C_1(d, \gamma)$.*

In other words, the number of $\gamma$-saturated $k$-rich hyperplanes cannot exceed

$$\begin{cases} 2C_1(d,\gamma)\frac{n^d}{k^{d+1}}, & \text{if } k \leq \sqrt{n}; \\ 2C_1(d,\gamma)\frac{n^{d-1}}{k^{d-1}}, & \text{if } \sqrt{n} \leq k. \end{cases}$$

For $d = 2$, this is exactly the Szemerédi–Trotter Theorem 2.1 with $C_1(2, \gamma) = C_0$. We prove Theorem 2.3 for $d = 3$ in Section 3, and in full generality in Section 4 below.

Our upper bound $O(n^d/k^{d+1})$ on the number of $k$-rich saturated hyperplanes, for $k \leq \sqrt{n}$, has already appeared in the second author's master thesis [17], advised (informally) by the first named author. Until now, however, it remained unnoticed that this bound is in fact sharp.

## 2.2 Not-Too-Degenerate Hyperplanes

We have seen that the Szemerédi–Trotter Theorem generalizes to saturated hyperplanes in any $\mathbb{R}^d$, $d \geq 3$. It is not easy to check, however, whether a $k$-rich hyperplane $H$ is $\gamma$-saturated or not: One should verify that $H \cap P$ spans at least $\gamma k^{d-1}$ distinct $(d-2)$-flats. In this subsection, we examine how Theorem 2.3 can be extended to hyperplanes satisfying a more straightforward condition.

**Definition 2.4** For integers $r, d \in \mathbb{N}$, $0 < r \le d$, we are given a point set $P$ and an $r$-flat $F^{(r)}$ in $\mathbb{R}^d$. We say that $F^{(r)}$ is *$\alpha$-degenerate*, $\alpha > 0$, if $F^{(r)} \cap P$ is non-empty and at most $\alpha \cdot |F^{(r)} \cap P|$ points of $F^{(r)} \cap P$ lie in an $(r-1)$-flat.

In particular, a plane $H$ is $\alpha$-degenerate if at most $\alpha|H \cap P|$ of its incident points are collinear. An $\alpha$-degenerate plane with $\alpha < 1$ must contain three affine independent points, while all points on a 1-degenerate plane may be collinear. The following theorem by Erdős and Beck implies that every $\alpha$-degenerate plane, $\alpha < 1$, is $\gamma$-saturated for a sufficiently small $\gamma > 0$.

**Theorem 2.5 (Erdős-Beck [3])** *Let $x$ and $k$ be integers with $0 < x \le k$. There is an absolute constant $c_0$ such that if $P$ is a set of $k$ points in the plane, at most $k - x$ of which are collinear, then $P$ spans at least $c_0 xk$ distinct lines.*

Applying Theorem 2.5 with $x = (1-\alpha)k$, we can conclude that every $\alpha$-degenerate plane ($\alpha < 1$) is, indeed, $c_0(1-\alpha)$-saturated.That is why in $\mathbb{R}^3$ we can also prove the following.

**Theorem 2.6** *For any $\alpha < 1$, there is a constant $C_2(\alpha) > 0$ such that for any set of $n$ points in $\mathbb{R}^3$, the number of $k$-rich $\alpha$-degenerate planes is at most*

$$C_2(\alpha)\left(\frac{n^3}{k^4} + \frac{n^2}{k^2}\right).$$

*This bound is best possible apart from the constant factor $C_2(\alpha) = C_1(3, c_0(1-\alpha))$.*

Unfortunately, Theorem 2.6 does not seem to directly generalize to higher-dimensional Euclidean spaces. The main reason for this is that the Erdős-Beck Theorem does not extend to $\mathbb{R}^d$, $d \ge 3$: Consider a point set located on two skew lines in $\mathbb{R}^3$, each containing $k/2$ points. This set is 1/2-degenerate but it spans only $k$ distinct planes.

In general, we know little about $\alpha$-degenerate point sets in higher dimensions. Gallai's theorem on ordinary lines was generalized to three-space by Motzkin [13] and to arbitrary dimensions $d \ge 3$ by Hansen [12]: If $P$ is a point set in $\mathbb{R}^d$ but not in a single hyperplane, then it spans a hyperplane $H$ such that $|H \cap P| - 1$ points in $H$ lie on a $(d-2)$-flat. Beyond this multidimensional Sylvester problem, the only known bound in this direction is Beck's "Two Extremes" Lemma.

**Lemma 2.7 (Beck [3])** *For any $r \ge 2$ there exist constants $\beta_r, \gamma_r \in (0, 1/2]$ such that, for any set $P$ of $k$ points in $\mathbb{R}^r$, at least one of the following holds.*

*(i) a hyperplane contains more than $\beta_r k$ points of $P$; or*

*(ii) the $r$-tuples of $P$ span at least $\gamma_r k^r$ distinct hyperplanes.*

*In other words, a $\beta_d$-degenerate hyperplane is $\gamma_d$-saturated.*

For $r = 2$, this is an obvious special case of the Erdős-Beck Theorem 2.5. For $r \ge 3$ and $\beta_r = 1/2$, no such $\gamma_r$ exists, as shown by the above example on two skew lines. The upper bound of Theorem 2.3 applies for such hyperplanes (and our lower bound construction also contains sufficiently many by-far-not-degenerate hyperplanes). Thus we have the following consequence.

**Theorem 2.8** *For any integer $d \geq 3$, there is a constant $C_3(d) > 0$ such that for any set of $n$ points in $\mathbb{R}^d$, the number of $k$-rich $\beta_{d-1}$-degenerate hyperplanes (where $\beta_{d-1} > 0$ is as in Lemma 2.7) is at most*

$$C_3(d)\left(\frac{n^d}{k^{d+1}} + \frac{n^{d-1}}{k^{d-1}}\right);$$

*this is best possible apart from the constant factor $C_3(d)$.* □

In other words, the number of $\beta_{d-1}$-degenerate $k$-rich hyperplanes in $\mathbb{R}^d$ is always bounded by $2C_3(d)n^d/k^{d+1}$ if $k \leq \sqrt{n}$; and by $2C_3(d)n^{d-1}/k^{d-1}$ if $\sqrt{n} \leq k$.

### 2.3 Spanned Hyperplanes

Using Theorem 2.8, we can give a short proof of a result of Agarwal and Aronov [1] about the number of $k$-rich hyperplanes spanned by a point set $P$ in $\mathbb{R}^d$, $d \geq 2$.

**Theorem 2.9** *For any $d \geq 2$, there is a constant $C_4(d) > 0$ such that for any set of $n$ points in $\mathbb{R}^d$, the number of $k$-rich hyperplanes containing $d$ affine independent points cannot exceed*

$$C_4(d)\left(\frac{n^d}{k^3} + \frac{n^{d-1}}{k}\right).$$

*Proof:* We proceed by induction on $d$. The base case, $d = 2$, is equivalent to the Szemerédi-Trotter Theorem 2.1. For any $d > 2$, we distinguish two classes of $k$-rich hyperplanes spanned by $P$: Those that contain more than $\beta_{d-1}k$ points in a $(d-2)$-dimensional flat, and those that do not. The number of ($\beta_{d-1}$-degenerate) hyperplanes in the latter class is bounded by $C_3(d)(n^d/k^{d+1} + n^{d-1}/k^{d-1})$ according to Theorem 2.8.

Consider the hyperplanes containing a $(\beta_{d-1}k)$-rich $(d-2)$-flat that contains $(d-1)$ affine independent points of $P$. If we project $P$ into a generic hyperplane, then every $(d-2)$-flat of $P$ projects into a $(d-2)$-flat in $\mathbb{R}^{d-1}$. We can apply induction: The number of $(\beta_{d-1}k)$-rich $(d-2)$-flats spanned by $P$ is at most $C_4(d-1)(n^{d-1}/(\beta_{d-1}k)^3 + n^{d-2}/(\beta_{d-1}k))$. Each such $(d-2)$-flat can lie in no more than $n$ hyperplanes spanned by $P$ in $\mathbb{R}^d$. Hence the number of hyperplanes containing a $(\beta_{d-1}k)$-rich $(d-2)$-flat is bounded by the required upper bound if $C_4(2) \geq C_0$ and $C_4(d) \geq 2\max\{C_4(d-1)/\beta_{d-1}^3, C_3(d)\}$ for $d \geq 3$.

This bound is tight apart from the constant factor by constructions of Edelsbrunner [6] (for $d = 3$), and Edelsbrunner and Haussler [8] ($d \geq 4$).

## 3 Upper and Lower Bounds in Three Dimensions

Consider a point set $P$ in $\mathbb{R}^3$ and a $\gamma$-saturated $k$-rich plane $S$. Since $S \cap P$ spans at least $\gamma k^2$ lines, some of the points in $S \cap P$ are incident to $\Omega(k)$ lines spanned by $P$. We call these points *strongly incident*:

**Definition 3.1** Given a point set $P \subset \mathbb{R}^3$ and a plane $S$, a point $p$ is *$w$-strongly incident* to $S$ if $S$ contains at least $w$ points $q_1, q_2, \ldots, q_w \in P \setminus \{p\}$ such that the lines $pq_i$, $i = 1, 2, \ldots, w$, are all distinct.

**Lemma 3.2** *For a point set $P$ and a $k$-rich $\gamma$-saturated plane $S$, at least $\gamma k/2$ points are $(\gamma k/2)$-strongly incident to $S$.*

*Proof:* Consider a $\gamma$-saturated plane $S$ and let $k = |S \cap P|$. Let us delete all points of $S \cap P$ which are incident to fewer than $\gamma k/2$ lines spanned by $S \cap P$. We deleted points from at most $k(\gamma \cdot k/2) = (\gamma k^2)/2$ lines spanned by $S \cap P$. From at least $\gamma k^2/2$ lines, no points were deleted, and so the remaining points of $P \cap S$ still span $\gamma k^2/2$ lines. Since every point in $S$ is incident to at most $k$ lines of $S$, there are at least $\gamma k/2$ points which are $(\gamma k/2)$-strongly incident to $S$.

## 3.1 Proof of Theorem 2.6

**Upper bound.** As mentioned after Theorem 2.5, every $\alpha$-degenerate plane is $\gamma := c_0(1-\alpha)$-saturated. Given a set $P$ of $n$ points in $\mathbb{R}^3$, let us denote by $s$ the number of $\gamma$-saturated $k$-rich planes. We apply Lemma 3.2 to each such plane $S$, and so we find $\gamma k/2$ points $(\gamma k/2)$-strongly incident to $S$. This gives a total of at least $s\gamma k/2$ strong point-plane incidences. Since we have $n$ points, at least one of them, say $p_1$, is $(\gamma k/2)$-strongly incident to at least $\gamma ks/(2n)$ planes.

Let $\pi$ be a plane not incident to $p_1$. We project every point of $P$ from $p_1$ to $\pi$. The image $P'$ consists of at most $n-1$ distinct points. $P' \subset \pi$ has the property that there are $\gamma ks/(2n)$ distinct lines, each containing at least $\gamma k/2$ distinct points of $P'$. By the Szemerédi–Trotter Theorem 2.1, we have

$$\frac{\gamma ks}{2n} \leq C_0\left(\frac{n^2}{(\gamma k/2)^3} + \frac{n}{\gamma k/2}\right).$$

We conclude that, indeed,

$$s \leq C_1(3,\gamma)\left(\frac{n^3}{k^4} + \frac{n^2}{k^2}\right),$$

for instance, for $C_1(3,\gamma) = 16 \cdot C_0/\gamma^4$.

**Lower bound.** We may assume that $k$ is even and $n$ is an integer multiple of $2k$. First, suppose that $k \leq \sqrt{n}$. We show a point set $P$ that admits at least $\frac{1}{4}(n^3/k^4)$ distinct $k$-rich 1/2-degenerate planes in $\mathbb{R}^3$. Let $P$ be the $\frac{k}{2} \times 2 \times \frac{n}{k}$ grid

$$P = \left\{(x,y,z) \in \mathbb{Z}^3 \ : \ 1 \leq x \leq \frac{k}{2},\ 1 \leq y \leq 2,\ 1 \leq z \leq \frac{n}{k}\right\}.$$

Both of the planes $y = 1$ or $y = 2$ contain many $(k/2)$-rich lines. Indeed, in either plane every line

$$z = m_i x + b_i, \qquad \text{where } 1 \leq m_i \leq \frac{n}{k^2},\ 1 \leq b_i \leq \frac{n}{2k},$$

passes through all points whose $x$-coordinate is in the range $x = 1, 2, \ldots, k/2$. Hence, for every $m_i \in \{1, \ldots, n/k^2\}$ and $b_i, b_j \in \{1, \ldots, n/(2k)\}$, the plane through the pair of parallel lines

$$\{y = 1, z = m_i x + b_i\} \quad \text{and} \quad \{y = 2, z = m_i x + b_j\}$$

intersects $P$ in $k$ points, located on the two lines. All these planes are 1/2-degenerate. The number of distinct planes is

$$(\#\text{ of } m_i) \cdot (\#\text{ of } b_i) \cdot (\#\text{ of } b_j) = \frac{n}{k^2} \cdot \left(\frac{n}{2k}\right)^2 = \frac{n^3}{4k^4}.$$

For $k \geq \sqrt{n}$, we use the same point set $P$. We find (again, for $\alpha = 1/2$) $n^2/k^2$ suitable $\alpha$-degenerate planes. For every pair $h_1, h_2 \in \{1, \ldots, n/k\}$, we consider the two *horizontal* lines

$$\{y = 1, z = h_1\} \quad \text{and} \quad \{y = 2, z = h_2\}.$$

Since both pass through $k/2$ points of $P$, the planes determined by these pairs of parallel lines still pass through $k$ points and they are also 1/2-degenerate. We have $(n/k)^2 = n^2/k^2$ such planes. □

**Remark 3.3** We note that for $k \leq \sqrt{n}$, we can find another construction with many not-too-degenerate $k$-rich planes. Without loss of generality we may assume that $k = t^2$ is a perfect square. Then it is not difficult to check that a $t \times t \times (n/t^2)$ grid allows for $\Theta(n^3/t^8) = \Theta(n^3/k^4)$ planes, each of which contains a $k = t \times t$ parallelogram lattice; e.g. the equations

$$z = a_i x + b_i y + c_i, \text{ where } 1 \leq a_i, b_i \leq \frac{n}{3t^3}, \text{ and } 1 \leq c_i \leq \frac{n}{3t^2},$$

describe such planes.

# 4 Upper and Lower Bounds in Higher Dimensions

## 4.1 Strong Incidences in $\mathbb{R}^d$

First we extend the notion of strong point-plane incidence to a strong incidence of an $(r-1)$-*tuple* of points to an $r$-flat.

**Definition 4.1** Given a point set $P \subset \mathbb{R}^d$ and an $r$-flat $F^{(r)}$, we say that an $(r-1)$-tuple $\{p_1, \ldots, p_{r-1}\} \subset F^{(r)} \cap P$ is $w$-*strongly incident* to $F^{(r)}$ if $F^{(r)}$ contains $w$ points $q_1, q_2, \ldots, q_w \in P \setminus \{p_1, \ldots, p_{r-1}\}$ such that the $(r-1)$-subflats determined by the $r$-tuples $\{q_i, p_1, \ldots, p_{r-1}\}$ (for $i = 1, 2, \ldots, w$) are all distinct. (In other words, $F^{(r)}$ contains at least $w$ distinct $(r-1)$-subflats spanned by $P$, each of which passes through the $(r-1)$-tuple $\{p_1, \ldots, p_{r-1}\}$ of $F^{(r)}$.)

As a special case, a (3-dimensional) hyperplane $H \subset \mathbb{R}^4$ is $w$-strongly incident to the pair of points $p_1, p_2 \in H \cap P$ if the points of $H \cap P$ determine at least $w$ distinct planes through $p_1 p_2$.

**Lemma 4.2** *For a point set $P$ and a $k$-rich $\gamma$-saturated $r$-flat $F^{(r)}$, at least $(\gamma/2)k^{r-1}$ distinct $(r-1)$-tuples of $P$ are $(\gamma k/2)$-strongly incident to $F^{(r)}$.*

*Proof:* Consider a $\gamma$-saturated $r$-flat $F^{(r)}$ and let $k = |F^{(r)} \cap P|$. For each of $\gamma k^r$ distinct $(r-1)$-subflats $F^{(r-1)}$ spanned by $F^{(r)} \cap P$, we select an affine independent $(r-1)$-tuple of $F^{(r-1)} \cap P$. (An $(r-1)$-tuple, of course, does not span $F^{(r-1)}$.) Some of these $(r-1)$-tuples may occur many times, since the total number of distinct $(r-1)$-tuples of $F^{(r)} \cap P$ is only $\binom{k}{r-1} < k^{r-1}$.

The multiplicity of the $(r-1)$-tuples which are *not* $(\gamma k/2)$-strongly incident to $F^{(r)}$ is less than $\gamma k/2$. They can contribute to at most

$$\binom{k}{r-1} \cdot \frac{\gamma k}{2} < \frac{\gamma k^r}{2}$$

$(r-1)$-subflats of $F^{(r)}$.

The remaining $\gamma k^r/2$ distinct $(r-1)$-subflats of $F^{(r)}$ correspond to $(r-1)$-tuples that are $(\gamma k/2)$-strongly incident to $F^{(r)}$. Since the multiplicity of any such $(r-1)$-tuple is at most $k$ (each of them needs one more point of $F^{(r)}$ to define an $(r-1)$-subflat) we conclude that $F^{(r)}$ has at least $(\gamma/2)k^{r-1}$ such $(r-1)$-tuples.

### 4.2 Proof of Theorem 2.3

**Upper bound.** Denote by $h$ the number of $k$-rich $\gamma$-saturated hyperplanes of $P$. We apply Lemma 4.2 to each such hyperplane with $r = d-1$. We find $h(\gamma/2)k^{d-2}$ distinct $(\gamma k/2)$-strong $(d-2)$-tuple—hyperplane incidences.

Since there are fewer than $n^{d-2}$ distinct $(d-2)$-tuples in $P$, at least one of them, say $\{p_1,\dots,p_{d-2}\}$ must be $(\gamma k/2)$-strongly incident to at least

$$\frac{\gamma}{2}\cdot\frac{hk^{d-2}}{n^{d-2}}$$

$k$-rich $\gamma$-saturated hyperplanes. The number of $(d-2)$-flats spanned by $P$ which contain the $(d-2)$-tuple $\{p_1,\dots,p_{d-2}\}$ cannot exceed $n-(d-2)<n$ (since each such flat, as usual, requires one more point from $P$).

Let $\pi$ be a plane (a 2-flat) that intersects all the $(d-2)$-flats incident to $\{p_1,\dots,p_{d-2}\}$ in distinct points. The intersection points give a set $T\subset\pi$ of size less than $n$. Each hyperplane $H$ which is $(\gamma k/2)$-strongly incident to $\{p_1,\dots,p_{d-2}\}$ passes through a line containing $\gamma k/2$ intersection points in $T$. Therefore, by the Szemerédi–Trotter Theorem 2.1, we have

$$\frac{\gamma}{2}\cdot\frac{hk^{d-2}}{n^{d-2}}\le C_0\cdot\left(\frac{n^2}{(\gamma k/2)^3}+\frac{n}{\gamma k/2}\right),$$

whence

$$h\le\frac{16C_0}{\gamma^4}\cdot\frac{n^d}{k^{d+1}}+\frac{4C_0}{\gamma^2}\cdot\frac{n^{d-1}}{k^{d-1}},$$

which is the required bound, e.g., if $C_1(d,\gamma):=16\cdot C_0/\gamma^4$.

**Lower bound.** We may assume that $k$ is a multiple of $d-1$, and $n$ is multiple of $d(d-1)2^{d-2}k$. First, suppose that $k\le\sqrt{n}$. We construct a point set for $\gamma = 1/(d-1)^{d-1}$ that admits many distinct $\gamma$-saturated $k$-rich hyperplanes (specifically, at least $c_d n^d/k^{d+1}$ hyperplanes, for a constant $c_d=(d-1)^2/(2^{d-2}d^d)$).

For positive integers $m$ and $M$, to be specified later, let $L$ be the $2\times 2\times\ldots\times 2\times m\times M$ grid

$$\begin{aligned}L \;=\; & \big\{(x_1,x_2,\dots,x_d): x_1,x_2,\dots,x_{d-2}\in\{0,1\},\\ & x_{d-1}\in\{1,\dots,m\}, x_d\in\{1,\dots,M\}\big\}.\end{aligned}$$

$L$ has $n=2^{d-2}mM$ points. We claim that each of the hyperplanes $H$ defined by the equations

$$x_d = a_1x_1+\ldots+a_{d-2}x_{d-2}+a_{d-1}x_{d-1}+b,$$

where

$$\begin{aligned}a_1,a_2,\dots,a_{d-2}&\in\left\{1,\dots,\frac{M}{d}\right\};\\ a_{d-1}&\in\left\{1,\dots,\frac{M}{md}\right\};\\ b&\in\left\{1,\dots,\frac{M}{d}\right\},\end{aligned}$$

contains $(d-1)$ "sufficiently independent" $m$-rich lines which, consequently, determine $m^{d-1}$ distinct $(d-2)$-subflats in $H$. Indeed, consider the planes (2-flats) $\pi_0, \pi_1, \ldots, \pi_{d-2}$ defined by

$$(x_1, \ldots, x_{d-2}) = \begin{cases} (0, \ldots, 0); & \text{and} \\ \mathbf{e}_i^{d-2} & \text{with } i = 1, 2, \ldots, d-2, \end{cases}$$

where $\mathbf{e}_i^{d-2}$ is a $d-2$-dimensional vector whose $i$-th coordinate is 1 and all other coordinates are 0. Each hyperplane $H$ intersects each $\pi_i$ in a straight line which passes through $m$ points of $L$. Specifically, $H$ intersects $\pi_i$ in a line through the points for which $x_{d-1} = 1, \ldots, m$ since these values satisfy

$$1 \le x_d \le (d-2) \cdot \frac{M}{d} + m \cdot \frac{M}{md} + \frac{M}{d} = M.$$

If we pick a point from each line $H \cap \pi_i$, then their $(d-1)$-tuple determine a $(d-2)$-subflat in $H$. We obtain distinct $(d-2)$-subflats for different choices due to the general position of the $\pi_i$-s. We conclude that, on the one hand, each $H$ is $m(d-1)$-rich and $(d-1)^{1-d}$-saturated because the cardinality of $H \cap L$ is

$$m^{d-1} = \frac{1}{(d-1)^{d-1}} \big(m(d-1)\big)^{d-1} = \gamma \cdot \big(m(d-1)\big)^{d-1}.$$

On the other hand, the number of the hyperplanes $H$ is the product of the number of possible values of $a_i$, $i = 1, 2, \ldots, d$, which is

$$\left(\frac{M}{d}\right)^{d-2} \cdot \frac{M}{md} \cdot \frac{M}{d} = \frac{1}{d^d} \cdot \frac{M^d}{m}.$$

Thus the grid $L$ has all the required properties if

$$m := \frac{k}{d-1} \qquad \text{and} \qquad M := \frac{d-1}{2^{d-1}} \cdot \frac{n}{k}.$$

For $\sqrt{n} \le k \le n/(d-1)$, (under the same divisibility assumptions on $k$ and $n$) we present a construction with $\frac{1}{2} n^{d-1}/k^{d-1}$ distinct $\gamma$-saturated $k$-rich hyperplanes for $\gamma = 1/(d-1)^{d-1}$.

First we select

$$m := (d-1)\frac{n}{k}$$

points $p_1, \ldots, p_m$ in the hyperplane $x_d = 0$ in general position. Then we consider the $m$ vertical lines through the $p_i$-s (i.e., straight lines parallel to the $x_d$-axis that pass through points $p_i$). We place

$$\frac{k}{d-1} - 1$$

new points on each such line, giving $k/(d-1)$ points on a line and a total of $n$ points altogether.

Now any system of $d-1$ of the $m$ vertical lines will determine a hyperplane $H$, distinct systems determine distinct hyperplanes. On the one hand, each such hyperplane contains $(d-1) \cdot k/(d-1) = k$ points and these points determine

$$\left(\frac{k}{d-1}\right)^{d-1} = \frac{1}{(d-1)^{d-1}} \cdot k^{d-1} = \gamma \cdot k^{d-1}$$

distinct $d-2$–subflats of $H$. On the other hand, since $m \geq k(d-1) \geq 2(d-1)$, the number of such hyperplanes is

$$\binom{m}{d-1} \geq \frac{(m/2)^{d-1}}{(d-1)!} = \frac{1}{2^{d-1}} \cdot \frac{(d-1)^{d-1}}{(d-1)!} \cdot \frac{n^{d-1}}{k^{d-1}} \geq \frac{n^{d-1}}{2k^{d-1}},$$

since the sequence $(r/2)^r/r!$ is known to be non-decreasing and equal to $1/2$ for $r = 1$ or 2. □

### 4.3 Some Consequences

From Theorem 2.3, we derive bounds on the number of distinct $d$-tuples of points lying in $\gamma$-saturated hyperplanes. These bounds generalize Beck's results [3] about the total number of point pairs lying on the at least $K$-rich or on the at most $M$-rich lines in the plane.

**Corollary 4.3** *Consider a point set $P$ in $\mathbb{R}^d$, $d \geq 3$, and let $K$ be an integer, $d \leq K \leq \sqrt{n}$. Consider all $\gamma$-saturated hyperplanes $H$ such that $K \leq |H \cap P| \leq \sqrt{n}$. These hyperplanes jointly contain at most $2^{d+1}C_1(d,\gamma)n^d/K$ distinct $d$-tuples of $P$. (Constant $C_1(d,\alpha)$ is from Theorem 2.3).*

*Proof:* For $i = 0, 1, 2, \ldots$, let $\mathcal{H}_i$ be the set of $\gamma$-saturated hyperplanes $H$ such that

$$2^i K \leq |H \cap P| < 2^{i+1}K.$$

By Theorem 2.3, we have

$$|\mathcal{H}_i| \leq C_1(d,\gamma)\frac{n^d}{\left(2^i K\right)^{d+1}},$$

if $2^i K \leq \sqrt{n}$. Since each such plane contains at most $\left(2^{i+1}K\right)^d$ distinct $d$-tuples of $P$, we derive the upper bound

$$C_1(d,\gamma)\frac{n^d}{\left(2^i K\right)^{d+1}}(2^{i+1}K)^d = C_1(d,\gamma)2^{d-i} \cdot \frac{n^d}{K}$$

for the number of $d$-tuples lying in hyperplanes of $\mathcal{H}_i$. By summing for *all* $i \geq 0$, we obtain the required bound of $2^{d+1}C_1(d,\gamma)n^d/K$.

**Corollary 4.4** *Consider a point set $P$ in $\mathbb{R}^d$, $d \geq 3$, and let $M$ be an integer, $\sqrt{n} \leq M \leq n$. Consider all $\gamma$-saturated hyperplanes $H$ such that $\sqrt{n} \leq |H \cap P| \leq M$. These hyperplanes jointly contain at most $2^d C_1(d,\gamma)Mn^{d-1}$ distinct $d$-tuples of $P$. (Constant $C_1(d,\gamma)$ is from Theorem 2.3).*

*Proof:* For $i = 0, 1, 2, \ldots$, let $\mathcal{H}_i$ be the set of $\gamma$-saturated hyperplanes $H$ such that

$$\frac{M}{2^{i+1}} < |H \cap P| \leq \frac{M}{2^i}.$$

By Theorem 2.3, we have

$$|\mathcal{H}_i| \leq C_1(d,\gamma)\frac{n^{d-1}}{\left(M/2^{i+1}\right)^{d-1}},$$

if $\sqrt{n} \leq M/2^{i+1}$. Since each such plane contains at most $(M/2^i)^d$ distinct $d$-tuples of $P$, we have the upper bound

$$C_1(d,\gamma)\frac{n^{d-1}}{\left(M/2^{i+1}\right)^{d-1}}(M/2^i)^d = C_1(d,\gamma)2^{d-i-1} \cdot Mn^{d-1}.$$

We obtain the required bound by summing these terms for *all* $i \geq 0$.

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