Distinct triangle areas in a planar point set

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March 8, 2007

Abstract

Erdős, Purdy, and Straus conjectured that the number of distinct (nonzero) areas of the triangles determined by n noncollinear points in the plane is at least $\lfloor \frac{n-1}{2} \rfloor$, which is attained for $\lceil n/2 \rceil$ and respectively $\lfloor n/2 \rfloor$ equally spaced points lying on two parallel lines. We show that this number is at least $\frac{17}{38}n - O(1) \approx 0.4473n$. The best previous bound, $(\sqrt{2}-1)n - O(1) \approx 0.4142n$, which dates back to 1982, follows from the combination of a result of Burton and Purdy [5] and Ungar's theorem [23] on the number of distinct directions determined by n noncollinear points in the plane.

1 Introduction

Let S be a finite set of points in the plane. Consider the (nondegenerate) triangles determined by triples of points of S. There are at most $\binom{n}{3}$ triangles, some of which may have the same area. Denote by $g(S)$ the number of *distinct (nonzero) areas* of the triangles determined by S. For every $n \in \mathbb{N}$, let $g(n)$ be the minimum of $g(S)$ over all sets S of n noncollinear points in the plane. The problem of finding $g(n)$ has a long history; the attention it has received is perhaps due to its simplicity and elegance, as well as to its connections to another fundamental problem in combinatorial geometry—that of finding the minimum number of directions spanned by n points in the plane. The problem of *distinct areas* is also similar in nature to a notoriously hard problem of *distinct distances*. It is listed for instance in the problem collection by Croft, Falconer, and Guy [6], and more recently by Braß, Moser, and Pach [3]; see also [12].

The first estimates on $g(n)$ were given in 1976 by Erdős and Purdy [10], who proved that

$$c_1 n^{3/4} \leq g(n) \leq c_2 n,$$

for some absolute constants $c_1, c_2 > 0$. The upper bound follows easily if we consider the points $(i,j) \in \mathbb{N}^2$ for $1 \leq i,j \leq \sqrt{n}$ and observe that every triangle area is a multiple of $\frac{1}{2}$ and bounded by $n/2$. A simple construction that consists of two sets of $\lceil n/2 \rceil$ and respectively $\lfloor n/2 \rfloor$ equally spaced points lying on two parallel lines was found by Burton and Purdy [5], and also by Straus [21]: It gives $\lfloor \frac{n-1}{2} \rfloor$ triangles of distinct areas.

In 1979, Burton and Purdy [5] obtained a linear lower bound, which follows from a linear bound on the number of directions determined by n noncollinear points in the plane. More precisely, denoting by $f(n)$ the minimum number of directions determined by n noncollinear points in the plane, they showed that

$$\left\lfloor \frac{n}{2} \right\rfloor \leq f(n) \leq 2 \left\lfloor \frac{n}{2} \right\rfloor .$$

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Using this result, an averaging argument of Burton and Purdy gave

$$0.32n \leq g(n) \leq \left\lfloor \frac{n-1}{2} \right\rfloor .$$

In 1982, Ungar proved a sharp bound

$$f(n) = 2 \left\lfloor \frac{n}{2} \right\rfloor \tag{1}$$

on the minimum number of directions determined by n noncollinear points, using a purely combinatorial approach of *allowable sequences* devised by Goodman and Pollack [14, 15]. A combination of Burton and Purdy's argument [5] with Ungar's theorem [23] immediately gives

$$(\sqrt{2}-1)n - O(1) \leq g(n) \leq \left\lfloor \frac{n-1}{2} \right\rfloor .$$

In this paper, we refine Burton and Purdy's averaging argument by applying yet one more time (and perhaps not for the last time) Ungar's technique on allowable sequences, and further improve the lower bound on distinct triangle areas.

Theorem 1 *The number of triangles of distinct areas determined by n noncollinear points in the plane is at least*

$$g(n) \geq \frac{17}{38}n - O(1) \approx 0.4473n.$$

In fact, we prove Theorem 1 in a stronger form: There are at least $17n/38 - O(1)$ triangles of distinct areas having *a common side*, in other words there are at least this many points of our set at distinct distances from the line determined by a pair of points in the set. One can draw here a parallel with the problem of distinct distances raised by Erdős in 1946: What is the minimum number of distinct distances $t(n)$ determined by n points in the plane? Erdős conjectured that $t(n) = \Omega(n/\sqrt{\log n})$, and moreover, that there is a point in the set which determines this many distinct distances to other points. In a sequence of recent breakthrough developments since 1997, all new lower bounds on $t(n)$ due to Székely [22], Solymosi and C. Tóth [20], and including the current best one due to Katz and Tardos [16], in fact give lower bounds on the maximum number of inter-point distances measured *from a single point*. For triangles areas in the plane, we have a similar phenomenon: By the argument of Burton and Purdy [5], every set S of n noncollinear points in the plane contains two distinct points $p, q \in S$ such that the points of S determine $\Omega(n)$ distinct distances to the line pq, therefore at least this many triangles with distinct areas. As mentioned above, our bound holds also in this stronger sense. A similar example is that of tetrahedra of distinct volumes determined by a set of n points in $\mathbb{R}^3$ (not all in the same plane): we have recently shown [8] that n points determine $\Omega(n)$ tetrahedra of distinct volumes, which share a common side. One exception to this phenomenon is the problem of distinct distances among vertices of a convex polygon, as the results of [1, 2, 7] show (see also [3]).

2 Proof of Theorem 1

Burton and Purdy's idea. We first review Burton and Purdy's argument [5]. Let S be a set of n noncollinear points in the plane, and let L denote the set of connecting lines (i.e., lines incident to at least 2 points of S). We may assume w.l.o.g. that there is no horizontal line in L. For a line $\ell \in L$, let

$\ell_1, \ell_2, \ldots, \ell_r \in L$ be all connecting lines parallel to ℓ (including ℓ) such that ℓ_i lies to the left of ℓ_{i+1} for $1 \leq i < r$. Let $k_i \geq 2$ denote the number of points along $\ell_i \in L$ for $i = 1, \ldots, r$. Let s be the number of *singleton* points of S not covered by any of $\ell_1, \ldots, \ell_r$. We clearly have $\sum_{i=1}^{r} k_i + s = n$. Taking any two points $p, q \in S$ on ℓ_1 or on ℓ_r, the triangles Δpqz_i have different areas for at least $r + \lceil s/2 \rceil - 1$ indices i, where z_i are either singleton points or points on different connecting lines lying all on the same side of pq. Therefore the number m of distinct areas satisfies

$$m \geq r + \lceil s/2 \rceil - 1.$$

The next step is selecting a suitable direction of connecting lines, more precisely, one with a small number of pairs of points, i.e., with a small value of $\sum_{i=1}^{r} \binom{k_i}{2}$. By Ungar's theorem, there is a direction corresponding to the lines $\ell_1, \ldots, \ell_r$, such that

$$\sum_{i=1}^{r} \binom{k_i}{2} \leq \binom{n}{2} \Big/ (n-1) = \frac{n}{2}.$$

After observing that $\sum_{i=1}^{r} \binom{k_i}{2}$ is minimal if the points on these r connecting lines are distributed as evenly as possible, Burton and Purdy derive a quadratic equation whose solution gives (using Ungar's theorem instead of their weaker bound of $\lfloor n/2 \rfloor$ on the number of directions) a lower bound of $m \geq (\sqrt{2} - 1)n - O(1) \approx 0.4142n$ on the number of distinct triangle areas. Detailed calculations show that a configuration attaining the Burton-Purdy bound should have $2 + \sqrt{2}$ points on each connecting line parallel to the certain direction (determined by at most $n/2$ pairs of points), a value which is certainly infeasible.

A tiny improvement. We first formulate a system of linear inequalities (the linear program (LP1) below). Unlike Burton and Purdy's quadratic equation, our linear program imposes an integrality condition on the number of points on each connecting line parallel to a specified direction; which leads to a tiny improvement ($5/12$ versus $\sqrt{2} - 1$). More important, our linear system paves the way for a more substantial improvement obtained by two linear programs with additional constraints (to be described later).

Assume that the connecting lines $\ell_1, \ell_2 \ldots, \ell_r \in L$ are vertical and contain at most $n/2$ point pairs (by Ungar's theorem). Every vertical line of L (passing through at least two points) is called a *regular line*. A regular line passing through *exactly* k points ($k \geq 2$) is called a k-line. We call a vertical line passing through exactly one point of S a *singleton line*.

Partition the n points of S as follows. Let s be a real number $0 \leq s < 1$ such that there are sn singleton points to the left of the leftmost regular line ℓ_1. Similarly, let tn be the number of singleton points to the right of ℓ_r, and let $a_1 n$ be the number of remaining singleton points. (See Figure 1.) For $k = 2, 3, \ldots, 8$, let $a_k n$ be the number of points on k-lines. Finally denote by a_9 the total number of points on regular lines with at least 9 points each. We have accounted for all points of S, hence we have

$$s + t + \sum_{k=1}^{9} a_k = 1.$$

Let $xn = \sum_{i=1}^{r} \binom{k_i}{2}$ be the total number of point pairs on vertical lines. Let en denote the number of distinct horizontal distances measured from the leftmost regular line ℓ_1 to its right: Consequently, there are en triangles with distinct areas having a common side along the leftmost regular line. Similarly, let fn denote the number of distinct horizontal distances measured from the rightmost regular line ℓ_r to its left. We can deduce lower bounds on e and f: Since $en \geq tn + a_1 n + a_2 n/2 + a_3 n/3 + \ldots + a_8 n/8 - 1$, we have

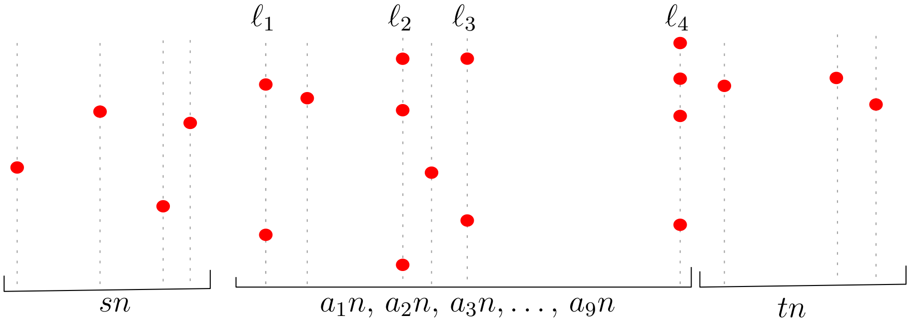

Figure 1: The orthogonal projection of a point set S in a direction determined by S.

$e \geq t + a_1 + a_2/2 + a_3/3 + \ldots + a_8/8 - 1/n$, and similarly, $f \geq s + a_1 + a_2/2 + a_3/3 + \ldots + a_8/8 - 1/n$. We can also give a lower bound for x in terms of the previous parameters. We have

$$x \geq \frac{1}{2}a_2 + \frac{2}{2}a_3 + \frac{3}{2}a_4 + \ldots + \frac{8}{2}a_9,$$

since if there are $a_k n$ points on k-lines, then the number of k-lines is $a_k n/k$, and each k-line contains $\binom{k}{2}$ vertical point pairs. Hence, there are $a_k n\binom{k}{2}/k = a_k n(k-1)/2$ pairs of points on k-lines, $k = 2, 3, \ldots, 8$. Similarly there are at least $\frac{8}{2}a_9 n$ pairs of points on lines incident to at least 9 points. Putting all of these equations and inequalities together, we formulate the following linear program.

$$\begin{array}{lll}
\text{minimize} & r & \text{(LP1)} \\
\text{subject to} & x \leq 0.5; & \\
& \left\{ \begin{array}{l}
s + t + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 = 1; \\
\frac{1}{2}a_2 + a_3 + \frac{3}{2}a_4 + 2a_5 + \frac{5}{2}a_6 + 3a_7 + \frac{7}{2}a_8 + 4a_9 \leq x; \\
t + a_1 + \frac{1}{2}a_2 + \frac{1}{3}a_3 + \frac{1}{4}a_4 + \frac{1}{5}a_5 + \frac{1}{6}a_6 + \frac{1}{7}a_7 + \frac{1}{8}a_8 - \frac{1}{n} \leq e; \\
s + a_1 + \frac{1}{2}a_2 + \frac{1}{3}a_3 + \frac{1}{4}a_4 + \frac{1}{5}a_5 + \frac{1}{6}a_6 + \frac{1}{7}a_7 + \frac{1}{8}a_8 - \frac{1}{n} \leq f; \\
e \leq r; \\
f \leq r;
\end{array} \right. & \\
& s, t, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, e, f, r, x \geq 0; &
\end{array}$$

The linear system (LP1) does not describe completely a point configuration (e.g., we do not make any distinction among k-lines for $k \geq 9$), but all these inequalities must hold if the variables correspond to a point set S. Let (LP1') be the linear program obtained from (LP1) by removing the two terms $\frac{1}{n}$, and let r be its solution. Since the constraints are linear, the term $\frac{1}{n}$ can only contribute a constant additive blow-up in the LP solution. That is, if r is the solution of (LP1'), the solution of (LP1) is $r - O(1/n)$. We can deduce that there are at least $rn - O(1)$ distinct triangle areas with a common side on either ℓ_1 or ℓ_r.

A solution to (LP1') is $r = 5/12 \approx 0.4166$, attained for $s = t = 1/4$, $a_3 = 1/2$, $a_1 = a_2 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = 0$, $e = f = 5/12$, and $x = 1/2$. That is, there are $n/6$ 3-lines in the middle, and $n/4$ singleton lines on each side, and $5n/12 - O(1)$ distinct areas measured from left or right. Another optimal solution that looks similar consists of $n/12$ 4-lines in the middle, and $n/3$ singleton lines on each side, for which the number of distinct areas is also $5n/12 - O(1)$.

Allowable sequences. We now give a very brief account on Ungar's technique (following [23]) and allowable sequences [12], as they are relevant to our proof. Allowable sequences occur in the context of transforming the permutation $1, 2, \ldots, n$ into the reverse permutation $n, n-1, \ldots, 1$ by going through a sequence of permutations. The operation between two consecutive permutations, called *move*, consists of inverting pairwise disjoint increasing strings. In a geometric context, each symbol corresponds to a point in the plane; each permutation is the left-to-right order in an orthogonal projections of the points on a directed line. The directed line is rotated around the origin, and a move occurs when the normal of this line coincides with a direction of a connecting line (a line in L). An example of a sequence arising in this way is $1(23)4(56)$, $13(246)5$, $(136)425$, $63(14)25$, $6(34)(125)$, $64(35)21$, $6(45)321$, and 654321. We have put parentheses around the increasing string (called *block*s) reversed at the next move. So each permutation with the blocks enclosed in parentheses describes also the next move.

Ungar's theorem states that for even n, going from $1, 2, \ldots, n$ to $n, n-1, \ldots, 1$ but *not* in one move, requires at least n moves (in other words, if every block reversed has fewer than n elements, at least n moves are needed). The general idea in the proof is that building up a long increasing block involves many moves required by dismantling other (possibly long) decreasing blocks formed at earlier moves, and vice versa. More precisely, the moves have the following properties.

(I) In one move, a decreasing string can get shorter by at most one element at each end.

(II) in one move, an increasing string can get longer by at most one element at each end.

For instance, the reason for (I) is that a move reverses increasing strings, and so only the first and the last elements of a decreasing string can be part of a block in a move. We refer the reader to [23] for more details. Properties (I) and (II) further imply that if a block B of size at least 3 is reversed in one move, then all but the two extreme elements of B must be singletons in the next move. Analogously, if a block B of size at least 3 is reversed in a move, then at least one of its elements is a singleton in the previous move.

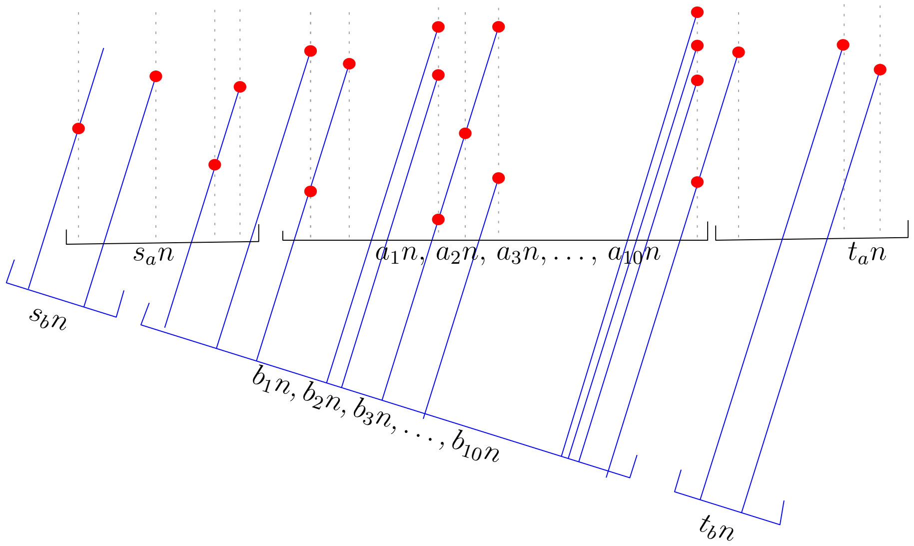

Figure 2: The orthogonal projection of a point set S in two consecutive directions, a and b, determined by S.

New bound. The idea for our new bound is the following. Recall that two optimal solutions of (LP1') we have seen have a similar structure: (A) $n/6$ 3-lines in the middle, and $n/4$ singleton lines on each side, or (B) $n/12$ 4-lines in the middle, and $n/3$ singleton lines on each side. Assume that there are two consecutive moves, π_1 and π_2, in an allowable sequence such that both look like (A) or (B). Notice that our observations regarding the blocks of size at least 3 imply that there cannot be two consecutive such moves, since the first move would force many singletons in the middle segment of π_2 (at least one for each block of π_1). This suggests that one of two consecutive directions of L must give a configuration where the solution of (LP1') is above $5/12$. We follow with the precise technical details in the proof of Theorem 1.

By Ungar's theorem, the average number of pairs determining the same direction is at most $n/2$, so there are two consecutive moves (corresponding to two consecutive directions of lines in L) parallel to at most n pairs of points. We introduce a similar notation as above for a single direction, but we distinguish the notation by indices a and b, respectively (e.g., $s_a n$ and $s_b n$ are the number of points which give singletons at the left side of the first and the second permutation, respectively). This time we count up to 9-lines (rather than 8-lines) and group together the k-lines for $k \geq 10$. We denote by $a_{10}n$ and $b_{10}n$ the total number of points on lines with at least 10 points each. By symmetry, we need to consider only two cases (instead of the four combinations of $s_a \lesseqgtr s_b$ and $t_a \lesseqgtr t_b$).

Case (i): $s_b \leq s_a$ and $t_b \leq t_a$.

Case (ii): $s_a \leq s_b$ and $t_b \geq t_a$.

We are lead to minimizing the following two linear programs (LP2i) and (LP2ii), where (LP2i) corresponds to Case (i) and (LP2ii) corresponds to Case (ii).

Case (i): $s_b \leq s_a$ and $t_b \leq t_a$. We formulate the linear program (LP2i) as follows. We repeat the constraints of (LP1) for both moves, and impose the constraint $x_a + x_b \leq 1$ since the total number of pairs for the two consecutive directions is at most n. We introduce two linear constraints to express $r = \max(r_a, r_b)$. Constraints (α) and (β) are crucial: Constraint (α) indicates that if in the first move, a block B of size at least 3 is reversed, then all but the two extreme elements of B must be singletons in the next move; constraint (β) specifies that each block B of size at least 3 which is reversed in the second move must contain an element which is a singleton in the first move (with the possible exception of two blocks that lie on the boundary of the singletons s_a and t_a).

Here is an example regarding constraint (β). Let π_1 and π_2 denote the two consecutive moves (each represented by pairwise disjoint blocks). The prefixes (resp., suffixes) of length s_b (resp., t_b) coincide, and are made of singletons. So each block of size at least 3 in the second move in between these common prefix and suffix strings (to be reversed in the second move) must pick up at least a singleton in a_1 from π_1 or must be made entirely up of singletons in the $(s_a - s_b)$ and $(t_a - t_b)$ segments of π_1 (except for at most two blocks crossing segment borders). For instance, if a move transforms permutation $\pi_1 = \ldots (47)(359) \ldots$ to $\pi_1' = \ldots 74953 \ldots$, then no triple (or other longer block) may be formed in the next move. But if there was a singleton in between, like in $\pi_1 = \ldots (47)6(359) \ldots$, then a triple may be formed in the next move: For instance, $\pi_2 = \ldots 7(469)53 \ldots$.

$$
\begin{array}{rl}
\text{minimize} & r \qquad\qquad\qquad\qquad \text{(LP2i)}\\
\text{subject to} & s_b \le s_a;\\
 & t_b \le t_a;\\
(\mathrm{LP1})_a & \left\{\begin{array}{l}
s_a + t_a + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} = 1;\\
\frac{1}{2}a_2 + a_3 + \frac{3}{2}a_4 + 2a_5 + \frac{5}{2}a_6 + 3a_7 + \frac{7}{2}a_8 + 4a_9 + \frac{9}{2}a_{10} \le x_a;\\
t_a + a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \frac{a_4}{4} + \frac{a_5}{5} + \frac{a_6}{6} + \frac{a_7}{7} + \frac{a_8}{8} + \frac{a_9}{9} - \frac{1}{n} \le e_a;\\
s_a + a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \frac{a_4}{4} + \frac{a_5}{5} + \frac{a_6}{6} + \frac{a_7}{7} + \frac{a_8}{8} + \frac{a_9}{9} - \frac{1}{n} \le f_a;\\
e_a \le r_a;\\
f_a \le r_a;
\end{array}\right.\\
(\mathrm{LP1})_b & \left\{\begin{array}{l}
s_b + t_b + b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9 + b_{10} = 1;\\
\frac{1}{2}b_2 + b_3 + \frac{3}{2}b_4 + 2b_5 + \frac{5}{2}b_6 + 3b_7 + \frac{7}{2}b_8 + 4b_9 + \frac{9}{2}b_{10} \le x_b;\\
t_b + b_1 + \frac{b_2}{2} + \frac{b_3}{3} + \frac{b_4}{4} + \frac{b_5}{5} + \frac{b_6}{6} + \frac{b_7}{7} + \frac{b_8}{8} + \frac{b_9}{9} - \frac{1}{n} \le e_b;\\
s_b + b_1 + \frac{b_2}{2} + \frac{b_3}{3} + \frac{b_4}{4} + \frac{b_5}{5} + \frac{b_6}{6} + \frac{b_7}{7} + \frac{b_8}{8} + \frac{b_9}{9} - \frac{1}{n} \le f_b;\\
e_b \le r_b;\\
f_b \le r_b;
\end{array}\right.\\
 & x_a + x_b \le 1;\\
 & r_a \le r;\\
 & r_b \le r;\\
(\alpha) & \frac{1}{3}a_3 + \frac{2}{4}a_4 + \frac{3}{5}a_5 + \frac{4}{6}a_6 + \frac{5}{7}a_7 + \frac{6}{8}a_8 + \frac{7}{9}a_9 + \frac{8}{10}a_{10} \le b_1;\\
(\beta) & b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9 + b_{10} - \frac{2}{n} \le 3a_1 + s_a - s_b + t_a - t_b;\\
 & s_a, t_a, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, e_a, f_a, r_a, x_a \ge 0;\\
 & s_b, t_b, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, e_b, f_b, r_b, x_b \ge 0;\\
 & r \ge 0;
\end{array}
$$

When we ignore the terms $O(\frac{1}{n})$, we get a new system (LP2i') with the following solution: $r = 17/38 \approx 0.4473$, attained for $s_a = t_a = 15/38$, $a_1 = a_2 = a_3 = 0$, $a_4 = 4/19$, $a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = 0$ for the first permutation, and $s_b = t_b = 3/38$, $b_1 = b_2 = 2/19$, $b_3 = 12/19$, $b_4 = b_5 = b_6 = b_7 = b_8 = b_9 = b_{10} = 0$ for the second permutation; also $x_a = 6/19$, $x_b = 13/19$, $e_a = f_a = r_a = e_b = f_b = r_b = 17/38$.

Case (ii): $s_b \le s_a$ and $t_b \ge t_a$. The linear program (LP2ii) is very similar to (LP2i). Besides the first two constraints, which are specific to this case, only constraints (γ) and (δ) are different: Constraint (γ) specifies that each block B of size at least 3 which is reversed in the second move must contain at least one singleton in the first move; constraint (δ) specifies the same thing when going back from the second permutation to the first one (by time reversibility).

$$
\begin{array}{rll}
\text{minimize} & r & \text{(LP2ii)} \\
\text{subject to} & s_b \le s_a; & \\
 & t_a \le t_b; & \\
(\mathrm{LP1})_a & \left\{ \begin{array}{l}
s_a + t_a + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} = 1; \\
\frac{1}{2}a_2 + a_3 + \frac{3}{2}a_4 + 2a_5 + \frac{5}{2}a_6 + 3a_7 + \frac{7}{2}a_8 + 4a_9 + \frac{9}{2}a_{10} \le x_a; \\
t_a + a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \frac{a_4}{4} + \frac{a_5}{5} + \frac{a_6}{6} + \frac{a_7}{7} + \frac{a_8}{8} + \frac{a_9}{9} - \frac{1}{n} \le e_a; \\
s_a + a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \frac{a_4}{4} + \frac{a_5}{5} + \frac{a_6}{6} + \frac{a_7}{7} + \frac{a_8}{8} + \frac{a_9}{9} - \frac{1}{n} \le f_a; \\
e_a \le r_a; \\
f_a \le r_a;
\end{array} \right. & \\
(\mathrm{LP1})_b & \left\{ \begin{array}{l}
s_b + t_b + b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9 + b_{10} = 1; \\
\frac{1}{2}b_2 + b_3 + \frac{3}{2}b_4 + 2b_5 + \frac{5}{2}b_6 + 3b_7 + \frac{7}{2}b_8 + 4b_9 + \frac{9}{2}b_{10} \le x_b; \\
t_b + b_1 + \frac{b_2}{2} + \frac{b_3}{3} + \frac{b_4}{4} + \frac{b_5}{5} + \frac{b_6}{6} + \frac{b_7}{7} + \frac{b_8}{8} + \frac{b_9}{9} - \frac{1}{n} \le e_b; \\
s_b + b_1 + \frac{b_2}{2} + \frac{b_3}{3} + \frac{b_4}{4} + \frac{b_5}{5} + \frac{b_6}{6} + \frac{b_7}{7} + \frac{b_8}{8} + \frac{b_9}{9} - \frac{1}{n} \le f_b; \\
e_b \le r_b; \\
f_b \le r_b;
\end{array} \right. & \\
 & x_a + x_b \le 1; & \\
 & r_a \le r; & \\
 & r_b \le r; & \\
(\gamma) & a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} - \dfrac{1}{n} \le 3b_1 + t_b - t_a; & \\
(\delta) & b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9 + b_{10} - \dfrac{1}{n} \le 3a_1 + s_a - s_b; & \\
 & s_a, t_a, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, e_a, f_a, r_a, x_a \ge 0; & \\
 & s_b, t_b, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, e_b, f_b, r_b, x_b \ge 0; & \\
 & r \ge 0; &
\end{array}
$$

When we ignore the terms $O(\frac{1}{n})$, we get a new system (LP2ii') with the following solution: $r = 25/54 \approx 0.4629$, attained for $s_a = t_a = 23/54$, $a_1 = 1/27$, $a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = 0$, $a_{10} = 1/9$, for the first permutation, and $s_b = t_b = 23/54$, $b_1 = 1/27$, $b_2 = b_3 = b_4 = b_5 = b_6 = b_7 = b_8 = b_9 = 0$, $b_{10} = 1/9$, for the second permutation; also $x_a = 1/2$, $x_b = 1/2$, $e_a = f_a = r_a = e_b = f_b = r_b = 25/54$.

Since the solution of (LP2i') is smaller than that of (LP2ii'), i.e., $17/38 < 25/54$, we conclude that there are always $\frac{17}{38}n - O(1) \approx 0.4473n$ triangles of distinct areas.

One may ask if the same result can be obtained using fewer variables in the LPs, or whether a better result can be obtained by increasing the number of variables in the LPs. The answer to both questions is negative.

3 Remarks

In 1982, Erdős, Purdy, and Straus [13] considered the generalization of the problem of distinct triangle areas to higher dimensions and posed the following:

Problem (Erdős, Purdy, and Straus). Let S be a set of n points in $\mathbb{R}^d$ not all in one hyperplane. What is the minimal number $g_d(n)$ of distinct volumes of nondegenerate simplices with vertices in S?

By taking d sets of about n/d equally spaced points on parallel lines through the vertices of a $(d-1)$-simplex, one gets $g_d(n) \leq \lfloor \frac{n-1}{d} \rfloor$. Erdős, Purdy, and Straus conjectured that equality holds at least for sufficiently large n (see also [6]). The first development in this old problem for higher dimensions is only very recent: for $d = 3$ we have shown that the tetrahedra determined by n points in $\mathbb{R}^3$, not all in a plane, have at least $\Omega(n)$ distinct volumes, which thereby confirms the conjecture in 3-space apart from the multiplicative constant [8].

We conclude with two problems on distinct triangle areas. The former is directly related to the original problem of distinct areas studied here, and appears to have been first raised by Erdős and Pach in the 1980s [17], while the latter appears to be new.

Given a planar point set S, consider the set L of connecting lines. A connecting line is called an *ordinary line* if it passes through exactly two points of S. By the well known Sylvester-Gallai theorem [18, 3], any finite set of noncollinear points in the plane determines an ordinary line. Consider now the set Θ of directions of lines in L. A direction $\theta \in \Theta$ is called an *ordinary direction* if all connecting lines of direction θ are ordinary lines.

Problem 1. Let S be a set of n noncollinear points in the plane. Is it true that apart from a finite set of values of n, Θ always contains an ordinary direction?

It should be clear that such a direction would be enough to prove the Erdős-Purdy-Strauss conjecture that S determines at least $\lfloor (n-1)/2 \rfloor$ distinct (nonzero) triangle areas — apart from a finite set of exceptions for n. Observe that $n = 7$ is such an exception, since the configuration of 7 points given by the three vertices of a triangle, the midpoints of its three sides, and the triangle center admits no ordinary direction.

Problem 2. Let S be a set of n noncollinear points in the plane. Is it true that each point $p \in S$ is the vertex of $\Omega(n)$ triangles of distinct areas determined by S? In other words, is there a constant $c > 0$ such that for every $p \in S$, the point set S determines at least cn triangles of distinct areas, all incident to p?

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