# Alternating Paths along Axis-parallel Segments*

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**Abstract**

It is shown that for a set $S$ of $n$ pairwise disjoint axis-parallel line segments in the plane there is a simple alternating path of length $\Omega(\sqrt{n})$. This bound is best possible in the worst case. In the special case that the $n$ pairwise disjoint axis-parallel line segments are *protruded* (that is, if the intersection point of the lines through every two nonparallel segments is not visible from both segments), there is a simple alternating path of length $n$.

Keywords: Alternating Path, Geometric Graph, Visibility

## 1 Introduction

For a set $S$ of pairwise disjoint line segments in the plane, the *segment endpoint visibility graph* $\text{Vis}(S)$ is the graph whose *vertices* are the segment endpoints and two vertices $u$ and $v$ are connected by an edge if $uv \in S$ (when the edge is called *segment edge*) or if the relative interior of the segment $uv$ does not intersect any segment of $S$ (*visibility edge*). The set $S$ of pairwise disjoint segments corresponds to a perfect matching $M(S)$ of $\text{Vis}(S)$. The segment endpoint visibility graph $\text{Vis}(S)$ has a natural straight line plane drawing, and so every path of the (abstract) graph $\text{Vis}(S)$ corresponds to a polygonal path in the plane. Segment endpoint visibility graphs have been extensively studied in the past. Researchers considered many aspects of these graphs including their computational complexity [20, 17, 23], the number of edges [22, 2], storage space [1, 10], and on-line updates [12].

In this paper, we focus on the longest alternating paths along disjoint segments. An *alternating path* for a set $S$ of pairwise disjoint line segments in the plane is a simple path $\pi$ in the segment endpoint visibility graph $\text{Vis}(S)$ where segment edges and visibility edges alternate. An alternating path $\pi$ for $S$ is *simple* if its natural planar embedding is a simple polygonal path in the plane, that is, if the planar embeddings of any two visibility edges of $\pi$ are disjoint. The length of an alternating path $\pi$ for $S$ is the number of segment edges along $\pi$. An alternating path is *Hamiltonian* if it passes though all segments of $S$. It is easy to see that not every set of disjoint segments admits an alternating *Hamiltonian* path: the smallest counterexample consist of six segments only (see e.g. [5]).

Urrutia [27, 28] and Bose [8] asked what is the maximal number $f(n)$ such that every set of $n$ pairwise disjoint line segments in the plane admits an alternating path of length $f(n)$. Hoffmann and

*Work on this paper was partially supported by National Science Foundation grants CCR-0049093 and IIS-0121562. A preliminary version of this paper has appeared in the *Proceedings of the 8th International Workshop on Algorithms and Data Structures (Ottawa, ON, 2003)*, vol. 2748 of Lecture Notes on Computer Science, Springer-Verlag, Berlin, 2003, pp. 389-400.

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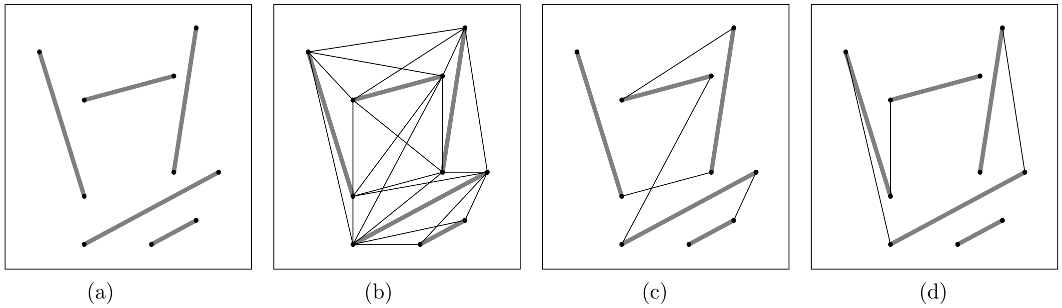


Figure 1: A set $S$ of 5 pairwise disjoint line segments (a). Their visibility graph $\mathrm{Vis}(G)$ (b). A nonsimple Hamiltonian alternating path (c). A simple alternating path of length 7 passing through 4 segments (d).

Tóth [14] have recently answered the question of Urrutia and Bose, and proved that $f(n) = \Theta(\log n)$. The upper bound construction [27, 14], for which every alternating path has $O(\log n)$ length, consists of line segments where the segment endpoints are in convex position and more than half of the segments lie on the convex hull $\mathrm{conv}(\bigcup S)$. Therefore, in that construction, $n$ segments have more than $n/4$ distinct orientations.

If the segments have a limited number of different slopes then a considerably longer alternating path may exist. This paper is the first attempt to show results in this direction. We show that if the segments have only two distinct orientations, then any $n$ disjoint segments admit an alternating path of length $O(\sqrt{n})$ and this bound is best possible apart from a constant factor.

**Theorem 1.**

*(i) For any $n$ pairwise disjoint axis-parallel segments in the plane, there is a simple alternating path of length at least $\sqrt{n/2}$.*

*(ii) For any $n \in \mathbb{N}$, there are $n$ pairwise disjoint axis-parallel segments in the plane such that the length of any alternating path is at most $4\sqrt{3n+4}+8$.*

We also consider line segments where the extensions of any two segments have disjoint relative interiors. The *extension* (in sign $\mathrm{ext}(s)$) of a segment $s \in S$ is the union of all line segment that contains $s$ but do not intersect the relative interior of any other segment of $S$. A set of disjoint line segments where the extensions have pairwise disjoint relative interiors is called a *protruded set*. It is known [25] that the protruded sets of segments are exactly those for which the minimum convex partitioning of the free space $\mathbb{E}^2 \setminus (\bigcup S)$ is unique. Any set $S$ of pairwise disjoint segments can be transformed into a *protruded* set by a simple *protruding* procedure, that is, by replacing every segment $s \in S$ inductively by a segment $\hat{s}$, $s \subseteq \hat{s} \subset \mathrm{ext}(s)$, such that $\hat{s}$ contains all intersections $\mathrm{ext}(s)$ with extensions of other segments.

It is not known, in general, whether every protruded set of segments admits a Hamiltonian alternating path. If the segments have only two distinct orientations, however, we can prove the following theorem.

**Theorem 2.** *Every finite set of pairwise disjoint axis-parallel line segments in the plane admits a simple Hamiltonian alternating path.*

Based on Theorem 2 and the *protruding* procedure, we can answer a question of Joe Mitchell about *1-2-alternating paths* on axis-parallel segments. Intuitively, consecutive input segments along an 1-2-alternating path are connected by two segments in the free space. A 1-2-alternating path for a set $S$ of disjoint line segments is defined as a simple polygonal path $p = (v_1, v_2, \ldots, v_k)$ such that $v_{3i-2}v_{3i-1} \in S$ and $v_{3i} \in \mathbb{E}^2 \setminus \bigcup S$ for $i = 1, 2, \ldots, \lfloor k/3 \rfloor$; and neither $v_{3i-1}v_{3i}$ nor $v_{3i}v_{3i+1}$ crosses properly any segment of $S$ for $i = 1, 2, \ldots, \lfloor (k-1)/3 \rfloor$. Note that a 1-2 alternating path does not correspond to a path in the segment endpoint visibility graph because the vertices $v_{3i}$, $i = 1, 2, \ldots, \lfloor k/3 \rfloor$, are not segment endpoints.

**Theorem 3.** *For every finite set $S$ of pairwise disjoint axis-parallel segments in the plane, there is a 1-2-alternating path through all segments of $S$.*

**Related problems.** A Ramsey-type problem asks for the maximal number $r(n)$ such that *any* set $S$ of $n$ pairwise disjoint line segments in the plane has a subset $S' \subseteq S$ that admits a Hamiltonian alternating path. The currently known best bounds, $r(n) = \Omega(n^{1/3})$ and $r(n) = O(\sqrt{n})$, are due to Pach and Pinchasi [21]. Note that a Hamiltonian alternating path for a subset $S'$, $S' \subset S$, is not necessarily an alternating path for $S$ since it may cross segments of $S \setminus S'$. An optimization problem asks for the longest alternating path for a given set $S$. The longest alternating path can be much longer than $f(n)$ in specific instances. We do not address either of these problems in this paper.

## 2 Constructing a long simple alternating path

Consider a set $S$ of $n$ pairwise disjoint axis-parallel line segments in the plane. We may assume without loss of generality that at least $n/2$ segments are horizontal. We let $H$, $H \subseteq S$, be the set of horizontal segments and let us denote the left and right endpoint of every $s_i \in H$ by $a_i$ and $b_i$, respectively.

For a point $p \in \mathbb{E}^2$, we denote by $x(p)$ and $y(p)$ the $x$- and $y$-coordinate of $p$. For two horizontal segments $s_1 = a_1b_1$ and $s_2 = a_2b_2$, we say that $s_2$ *cover*s $s_1$ (in sign, $s_1 \prec s_2$) if $x(a_1) < x(b_2)$ and $y(a_1) < y(b_2)$. The relation $\prec$ is cycle-free because it orders the horizontal segments according to their $y$-coordinates. It is not necessarily transitive, and so we define its transitive closure. We say that $s < t$ if and only if there is a sequence $(s = s_1, s_2, \ldots, s_r = t)$ in $H$ such that $s_{i+1}$ covers $s_i$, for $i = 1, 2, \ldots, r-1$ (Fig. 2). Thus, $(H, <)$ is a partially ordered set on the horizontal segments. Similar partial orders were previously used by Tamassia and Tollis [24] and by G. Tóth [26].

According to Dilworth's Theorem [9], the poset $(H, <)$ of size at least $n/2$ contains either a chain or an anti-chain of size $\sqrt{n/2}$. The segments of a chain or an anti-chain do not necessarily form a simple alternating path. The following lemma combined with Dilworth's Theorem immediately implies our lower bound, Theorem 1(i).

**Lemma 1.**

*(a) All segments in a chain of the poset $(H, <)$ lie on a simple alternating path for $S$.*

*(b) All segments in an anti-chain of the poset $(H, <)$ lie on a simple alternating path for $S$.*

The proof of Lemma 1 is postponed to Subsection 2.4. The next three subsections pave the way to this proof. We describe a recursive algorithm to build an alternating path passing through

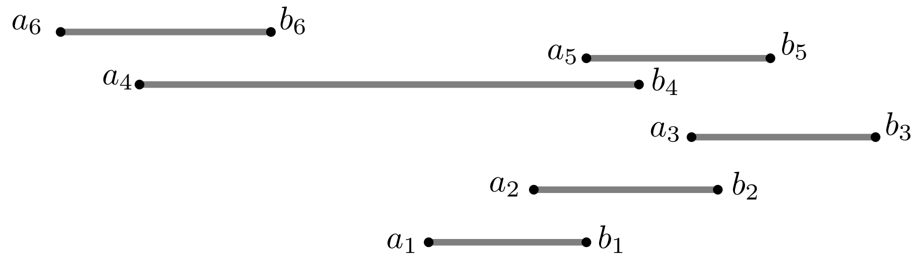



Figure 2: Six horizontal segments. A maximal chain $a_1b_1 \prec a_2b_2 \prec a_4b_4 \prec a_6b_6$ and a maximal anti-chain $(a_3b_3, a_6b_6)$. Note that $a_1b_1 < a_6b_6$ but $a_1b_1 \not\prec a_6b_6$.

all the segments of a chain or an anti-chain. The algorithm starts with an initial polygonal path $\gamma$, which is not necessarily a simple alternating path but passes through every segment of the chain or the anti-chain, and intersects all other input segments in at most one point. Then we apply successively a simple operation, called `Expand`, which modifies the path locally to pass along that segment if a path has one common point with a segment.

### 2.1 Partially monotone curves

Consider a *directed curve* $\pi = \{\pi(t) : t \in [0,1]\}$ in the plane. For two parameters $t_1$ and $t_2$, $0 \le t_1 \le t_2 \le 1$, let $\pi(t_1,t_2)$ denote the portion of $\pi$ from $\pi(t_1)$ to $\pi(t_2)$. We say that $\pi$ is *monotone increasing* (resp., *decreasing*) if for any two parameters $0 \le t_1 \le t_2 \le 1$, we have $x(\pi(t_1)) \le x(\pi(t_2))$ and $y(\pi(t_1)) \le y(\pi(t_2))$ (resp., $y(\pi(t_1)) \ge y(\pi(t_2))$). Similarly, $\pi$ is *strictly* monotone increasing (resp., decreasing) if for any $0 \le t_1 < t_2 \le 1$, we have $x(\pi(t_1)) < x(\pi(t_2))$ and $y(\pi(t_1)) < y(\pi(t_2))$ (resp., $y(\pi(t_1)) > y(\pi(t_2))$). A polygonal curve $v_1v_2 \ldots v_k$ is a directed curve obtained by the concatenation of the directed segments $\overrightarrow{v_1v_2}, \overrightarrow{v_2v_3}, \ldots, \overrightarrow{v_{k-1}v_k}$ with an arbitrary parameterizations.

Consider a set $S$ of pairwise disjoint axis-parallel segments in the plane and let $B$ be their axis-aligned bounding box. Let $a_0$ denote the lower left corner of $B$ and let $b_\infty$ be the upper right corner of $B$. Consider a curve $\gamma$ satisfying the following three invariants, which are crucial in our argument.

**Definition 1.** *If a curve $\gamma$ is called* partially monotone *with respect to a set $S$ of pairwise disjoint line segments in a bounding box $B$ if it satisfies the following conditions.*

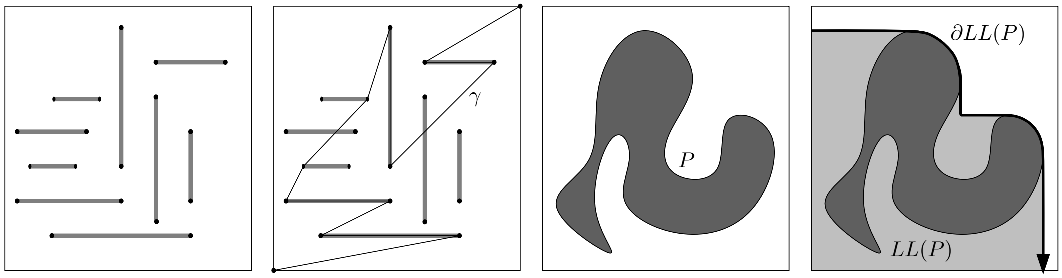
*(a) $\gamma : [0,1] \to B$ is a polygonal curve from $\gamma(0) = a_0$ to $\gamma(1) = b_\infty$, lying entirely in $B$.*

*(b) Every intermediate vertex of $\gamma$ is a segment endpoint of $S$.*

*(c) Every portion of $\gamma$ which does not lie along a segment of $S$ is monotone increasing.*

*(d) $\gamma$ passes through every segment of $S$ at most once. decreasing.*

For instance a curve along the directed segment $\overrightarrow{a_0b_\infty}$ is partially monotone w.r.t. $S$; and Fig. 3(b) depicts another example. We point out here a few important properties of partially monotone curves. We define the *lower-left hull* (for short, *LL-hull*) of a point set $P$ in a bounding




(a) (b) (c) (d)

Figure 3: A set $S$ of 7 pairwise disjoint axis-parallel line segments (a). A partially monotone curve $\gamma$ w.r.t. $S$ in (b). A set $P \subset B$ in (c), and $LL(P)$ in (d).

box $B$. The LL-hull of a set $P$, $P \subseteq B$, is the set of all points $q \in B$ for which there is a point $p \in P$ in the upper right quadrant of $q$. Formally, $LL(P) = \{q \in B \mid \exists p \in P : y(q) \leq y(p)$ and $x(q) \leq x(p)\}$.

Note that for any set $P \subset B$, the set $LL(P)$ is simply connected and so its boundary is a simple closed curve. The boundary of $LL(P)$ contains a connected portion of the left and the lower side of the bounding box $B$. Let $\partial LL(P)$ denote the directed curve along the boundary of $LL(P)$ from the left to the lower side of $B$. Observe that $\partial LL(P)$ is always a monotone decreasing curve. If $\partial LL$ contains a strictly monotone decreasing portion $\eta$, then $\eta \subseteq P$.

Let $A(t) := LL(\gamma(0,t))$, the lower-left hull of a prefix curve of $\gamma$ up to point $\gamma(t)$. If $\gamma$ is partially monotone w.r.t $S$, then $\partial A(t) = \partial LL(\gamma(0,t))$ is rectilinear because it is monotone decreasing but none of its portions is strictly monotone decreasing. We show that $\gamma$ never enters the interior of the LL-hull of its prefix curve.

**Lemma 2.** *If $\gamma$ is a partially monotone curve w.r.t. a set $S$ of pairwise disjoint axis-parallel segments, then $\gamma(t) \in \partial A(\gamma(t))$ for all $t$, $0 \leq t \leq 1$.*

*Proof.* We argue by contradiction. Let $\gamma(t_1) = p_1$ be the first point along $\gamma$ such that there is a line segment $p_1p_2 \subset \gamma$ where $p_1 \in \partial(A(t_1))$ and $p_1p_2 \setminus \{p_1\} \subset \text{int}(A(t_1))$. The segment $p_1p_2 \subset \gamma$ cannot lie along a monotone increasing portion of $\gamma$, because that would imply $p_1 \in LL(p_2)$ and $p_1 \in \text{int}(A(t_1))$. By Definition 1(c), $p_1p_2$ lies along a segment $s \in S$. We may suppose, w.l.o.g., that $s \in S$ is horizontal.

Since $\gamma$ enters $\text{int}(A(t_1))$ along a horizontal segment at $p_1$, there is a point in $\gamma(0,t_1)$ strictly above $p_1$. Let $t_0$, $0 < t_0 < t_1$ be the minimum value where $p_0 = \gamma(t_0)$ is such a point that $x(p_0) = x(p_1)$ and $y(p_0) > y(p_1)$. The vertical segment $p_0p_1$ remains on the boundary of $A(t)$ for every $t$, $t_0 \leq t \leq t_1$, since we assume that $p_1 \notin \text{int}(A(t))$.

Recall that $\gamma(t) \in \partial A(t)$ for all $0 \leq t \leq t_1$, and so $\gamma(t_0,t_1)$ must pass though the directed segment $\overrightarrow{p_0p_1}$, which is monotone decreasing. The last portion of $\overrightarrow{p_0p_1}$ cannot lie along a segment of $S$ because $p_1$ lies on the horizontal segment $s \in S$ and the segments of $S$ are pairwise disjoint. This contradicts Definition 1(c). We conclude that $\gamma(t) \in \partial A(t)$ for all $0 \leq t \leq 1$. □

**Lemma 3.** *Every partially monotone curve $\gamma$ w.r.t. a set $S$ of pairwise disjoint axis-parallel segments is a* simple *polygonal curve.*

*Proof.* Suppose that $\gamma$ passes through the same point twice, that is, there are $0 \leq t_1 < t_2 \leq 1$ such that $\gamma(t_1) = \gamma(t_2) = p$ and $\gamma(t) \neq p$ for any $t$, $t_1 < t < t_2$. The curve $\gamma(t_1, t_2)$ cannot enter the interior of $LL(p)$ by Lemma 2. Furthermore, $p$ must lie on the boundary $\partial A(t)$ for every $t$, $t_1 \leq t \leq t_2$. This implies that the curve $\gamma(t_1, t_2)$ cannot enter the lower left and upper right quadrants of $p$.

We may assume w.l.o.g. that $\gamma(t_1, t_2)$ lies in the upper left quadrant of $p$. Since the last portion of $\gamma(t_1, t_2)$ must be monotone decreasing, by Definition 1(c) it returns to $p$ along a segment $s \in S$. By Definition 1(d), it cannot leave $p$ along the same segment $s$. So the initial portion of $\gamma(t_1, t_2)$ is monotone increasing, which in the upper left quadrant of $p$ implies that it is vertical. In turn, $s$ has to be horizontal. So there is a point $p' \in \gamma(t_1, t_2)$ such that $x(p') = x(p_1)$ and $y(p') > y(p_1)$, and so the left endpoint of $s$ is in $\text{int}(LL(p'))$, contradicting Lemma 2. This proves Lemma 3. □

By Lemma 3, every partially monotone curve $\gamma$ partitions the bounding box $B$ into two regions. One region is adjacent to the left side of $B$ and the other to the right side of $B$. We call the closures of these two regions the *left* and the *right side* of $\gamma$, respectively.

**Lemma 4.** *If a curve $\gamma$ is partially monotone w.r.t. a set $S$ of pairwise disjoint axis-parallel segments, then the following two statements hold.*

*(i) For every segment $s \in S$, the intersection $\gamma \cap s$ is empty, one point, or $s$.*

*(ii) If a segment $s \in S$ lies on the left (right) side of $\gamma$ and has one common point with $\gamma$, then $t \cap \gamma$ is the right or lower (left or upper) endpoint of $s$.*

*Proof.* By Definition 1 (b), $\gamma$ has no vertices in the relative interior of any segment of $S$. This immediately implies that the intersection $\gamma \cap s$ is empty, a finite number of points, or the entire segment $s$. It suffices to show that a finite number of intersection points means a single intersection point. Suppose that there is a prefix $\gamma(0, t_1)$ of $\gamma$ such that the intersection $\gamma(0, t_1) \cap s$ with a segment $s \in S$ consists of exactly two points. We denote these points by $p_0 = \gamma(t_0)$ and $p_1 = \gamma(t_1)$.

We may assume w.l.o.g. that $s$ is horizontal, and let $\ell(s)$ be the line passing through $s$. By Definition 1 (c-d), the portions of $\gamma(t_0, t_1)$ directly following $p_0$ and preceding $p_1$ are monotone increasing. By Lemma 2, the curve $\gamma(t_0, t_1)$ cannot enter the interior of $A(t_0)$, and by assumption, it is disjoint from $s$. This implies that $p_1 \in s$ is either the left endpoint of $s$ or it lies to the right of $p_0$. First assume that $p_1$ is the left endpoint of $s$. In this case, there must be a point $p'\gamma(t_0, t_1)$ with $x(p') = x(p_0)$ and $y(p') > y(p_0)$. Since $p_1 \in \text{int}(LL(p'))$, curve $\gamma$ cannot reach $p_1$. Next assume that $p_1$ is to the right of $p_0$. There must be a point $p''$ with $x(p_0) < x(p'')$ and $y(p_0) = y(p'')$. Since the portion of $\gamma$ preceding $p_1$ is monotone increasing it lies in $\text{int}(LL(p''))$, and so $\gamma$ cannot reach $p_1$. this proves part (i) of Lemma 4. Finally, part (iii) of Lemma 4 follows from the fact that the common point $s \cap \gamma$ of $s$ and $\gamma$ lies on a monotone increasing portion of $\gamma$. □

## 2.2 The `Expand` operation

In this subsection, we define an operation `Expand`, originally introduced by Hoffmann and Tóth [15] for constructing a Hamiltonian circuit in $\text{Vis}(S)$)). We then prove here a simple new property of this operation. We start by defining a shortest path homotopic to a two-segment path in the free space of a set of line segments.

**Definition 2.** *Let $v_1$, $v_2$, and $v_3$ be three points in the plane and assume that the simple polygonal path $v_1v_2v_3$ does not intersect the relative interior of any segment of $S$. The* convex arc $\mathrm{carc}_S(v_1, v_2, v_3)$ *is the shortest polygonal path from $v_1$ to $v_3$ homotopic to $v_1v_2v_3$ in the complement of the line segments $\mathbb{E}^2 \setminus (\bigcup S)$.*

We note that if $v_1, v_2$, and $v_3$ are not collinear and $v_2$ is not a segment endpoint, then $\mathrm{carc}_S(v_1, v_2, v_3) \cup v_3v_2v_1$ is a *pseudo-triangle* where all internal vertices of $\mathrm{carc}_S(v_1, v_2, v_3)$ are reflex. Based on the definition of convex arcs, we can now define operation `Expand`, which replaces a segment of a polygonal path by a longer path.

**Operation 1.** *Expand$_S(\pi, va, u)$ (see Fig. 4).*
*Input*: a directed polygonal path $\pi$; a point $v \in \pi$ and a point $a \notin \pi$ such that $\pi$ and the segment $va$ partition the full angle about $v$ into three convex angular domains; an orientation $u \in \{-, +\}$; and a set $S$ of disjoint line segments in the plane.
*Operation*: Let $v^-$ and $v^+$ denote the vertices of $\pi$ preceding and following $v$, respectively. Obtain $\pi'$ from $\pi$ by replacing the edge $vv^u$ by the path $va \cup \mathrm{carc}_S(a, v, v^u)$.
*Output: $\pi'$.*

**Proposition 5.** *If a two-segment curve $v_1v_2v_3$ is monotone increasing and $\angle v_1v_2v_3 < 180°$ (resp., $\angle v_3v_2v_1 < 180°$), then every segment of the path $\mathrm{carc}_S(v_1, v_2, v_3)$ is also monotone increasing (decreasing) and $\mathrm{carc}_S(v_1, v_2, v_3)$ contains the right (left) endpoints of* horizontal *segments of $S$ and lower (upper) endpoints of* vertical *segments of $S$.* □

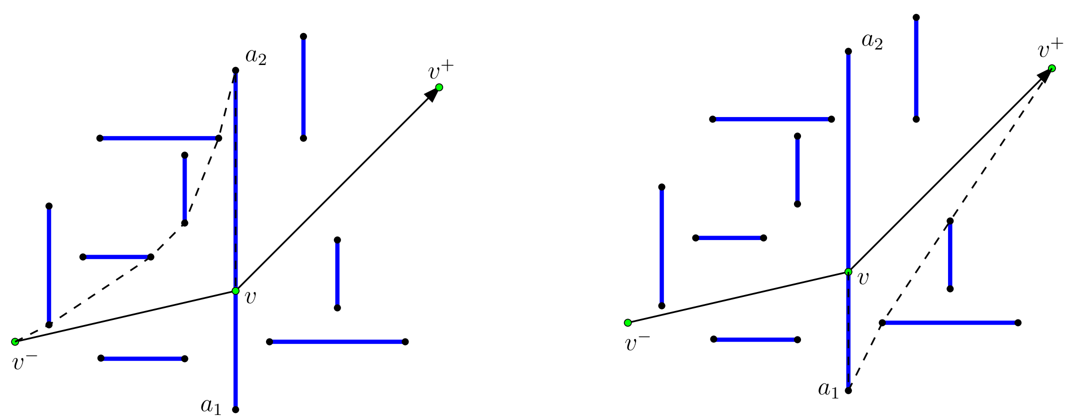



Figure 4: $\texttt{Expand}_S(\pi, va_1, -)$ on the left and $\texttt{Expand}_S(\pi, va_2, +)$ on the right.

### 2.3 Expanding a simple alternating path

We use operation `Expand` to transform a partially monotone curve into a partially monotone simple alternating path.

**Lemma 6.** *For a set $S$ of pairwise disjoint axis-parallel line segments in the plane and a partially monotone curve $\gamma$ w.r.t. $S$, there is a partially monotone simple alternating path $\gamma'$ which passes through every segment of $S$ that $\gamma$ passes through.*

*Proof.* We modify the path $\gamma$ by applying `Expand` successively in two phases. We show that our operations maintain a partially monotone curve w.r.t. $S$ throughout both phases. Lemma 4 allows three possible ways how $\gamma$ may intersect a segment $s \in S$: $\gamma$ can pass through $s$, it can pass through an endpoint of $s$ without passing through $s$, or it can cross $s$ (i.e., $\gamma$ has no vertex incident to $s$ and it intersects it in a single point). After the first phase, $\gamma$ does not cross any segment of $S$; and after the second phase it also does not pass through any segment endpoint without passing through the entire segment, and so it is a simple alternating path.

**First phase.** In the first phase, we modify $\gamma$ until we obtain a simple polygonal path that does not cross any line segment of $S$. Let $S(\gamma)$ denote the set of segments of $S$ crossed by $\gamma$. We repeat the following step until $S(\gamma)$ is empty. Consider the first segment $ab \in S$ crossed by $\gamma$. Let $x = ab \cap \gamma$ and let us assume that $a$ is the lower (left) endpoint and $b$ is the upper (right) endpoint of a vertical (horizontal) segment $ab$. We modify $\gamma$ by two operations. Let $\gamma := \texttt{Expand}_{S \setminus S(\gamma)}(\gamma, xa, +)$ and then let $\gamma := \texttt{Expand}_{S \setminus S(\gamma)}(\gamma, xb, -)$. As a result of the two `Expand` operations, $ab \cap \gamma = ab$. By Proposition 5, we obtain a partially monotone curve w.r.t. $S \setminus S(\gamma)$, which is also a partially monotone curve w.r.t. $S$. By Lemma 4 (i), every segment of $S(\gamma) \setminus \{ab\}$ is crossed at most once. The set $S(\gamma)$ of segments crossed by $\gamma$ decreases by at least one. Therefore, the first phase terminates in $O(n)$ steps.

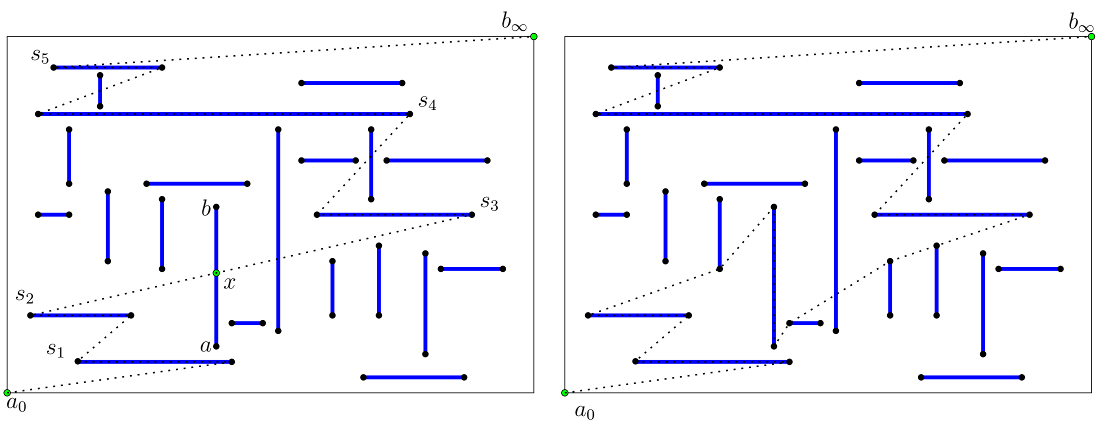



Figure 5: The path $\gamma$ in initial form (left), and after the first step of phase 1 (right).

**Second phase.** We expand recursively $\gamma$ into an alternating path. In this phase, we maintain a partially monotone curve w.r.t. $S$ with the property that $\gamma$ does not cross any segment of $S$. Consider the segments of $S$ whose one endpoint lies on $\gamma$, but which do not lie along $\gamma$. Let $a$ be the first vertex along $\gamma$ such that $ab = s \in S$ but $ab \not\subset \gamma$. We modify $\gamma$ to include $s$ and visit the endpoint $b$ as well.

- If $ab$ is vertical and lies on the left side of $\gamma$, or if $ab$ is horizontal and lies on the right side of $\gamma$, then apply $\texttt{Expand}_S(\gamma, ab, -)$.
- If $ab$ is vertical and lies on the right side of $\gamma$, or if $ab$ is horizontal and lies on the left side of $\gamma$, then apply $\texttt{Expand}_S(\gamma, ab, +)$.

We have set the orientation $u$ in every call of operation `Expand` such that $\angle baa^u < 180°$. Therefore, by Lemma 4 (i), we obtain a partially monotone curve w.r.t. $S$, and $\gamma$ does not cross

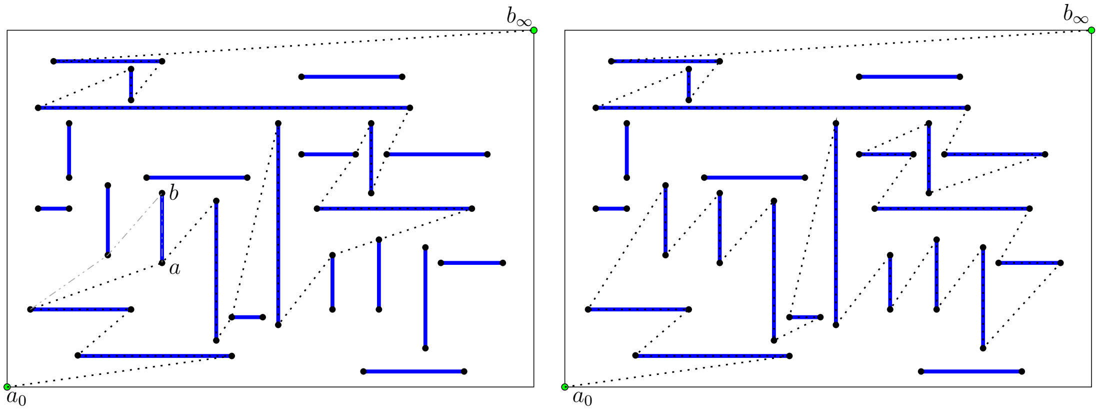



Figure 6: $\gamma$ at the end of phase 1 (left) and the output alternating path (right).

any segment of $S$. In every step of the second phase, the number of vertices of $\gamma$ increases by at least one, and so the phase terminates in $O(n)$ steps. If every segment of $S$ that intersects $\gamma$ actually lies along $\gamma$, then $\gamma$ is an alternating path (after removing the first and last edges incident to $a_0$ and $b_\infty$). The resulting alternating path passes through all segments the initial curve $\gamma$ passed through, as required. □

### 2.4 Construction of an alternating path

Consider a maximal chain in the partially ordered set $(H, <)$, that is, a sequence $s_1, s_2, \ldots, s_r$ of horizontal segments such that for every $i = 1, \ldots r-1$ we have $s_i < s_{i+1}$. Since there is no segment $t$ with $s_i < t < s_{i+1}$, we also know that $s_i \prec s_{i+1}$. Let us denote the left and right endpoint of $s_i$ by $a_i$ and $b_i$, resp., for $i = 1, 2, \ldots, r$; and let $a_0$ and $b_\infty$ be the lower left and upper right corners of the axis-aligned bounding box $B$ of $S$.

We define a simple polygonal path $\gamma$ from $a_1$ to $b_\infty$ which passes through the segments $s_1, s_2, \ldots, s_r$, but which is not necessarily alternating. We define the curve $\gamma = a_0b_1 \cup b_1a_1 \cup a_1b_2 \cup \ldots \cup b_ra_r \cup a_rb_\infty$ (Fig. 5, left). Observe that $\gamma$ is partially monotone w.r.t. $S$. By Lemma 6, it can be expanded into a simple alternating path such that it passes through $s_1, s_2, \ldots, s_r$ (Fig. 6, right).

Now consider an anti-chain $A = \{s_1, s_2, \ldots, s_r\}$ in the partially ordered set $(H, <)$. Let us denote the left (right) endpoint of $s_i$ by $a_i$ ($b_i$) for $i = 1, 2, \ldots, r$, and let $c_0$ and $d_\infty$ be the upper left and lower right corners of $B$. Notice that $s_i \not> s_j$ and $s_i \not< s_j$ implies that $s_i$ and $s_j$ are (weakly) separated by a vertical line, and so we can order the segments of $A$ according to their $x$-coordinates. We suppose that $s_1, s_2, \ldots, s_r$ are labeled in this order.

The directed segment $\overrightarrow{b_ia_{i+1}}$, $i = 1, 2, \ldots, r-1$, is monotone decreasing, otherwise $s_i < s_{i+1}$. We define the curve $\gamma = c_0a_1 \cup a_1b_1 \cup b_1a_2 \cup \ldots, a_rb_r \cup b_rd_\infty$ (Fig. 7, left). Thus, $\gamma$ is monotone decreasing. Reflect the input segments and $\gamma$ to the $x$ axis. The reflection exchanges the upper and lower sides of $B$, it also exchanges $c_0$ with $a_0$ and $d_\infty$ with $b_\infty$. It transforms $\gamma$ into a partially monotone curve w.r.t. $S$. By Lemma 6, it can be expanded into a simple alternating path such that it passes through $s_1, s_2, \ldots, s_r$ (Fig. 7, right). This completes the proof of Lemma 1.

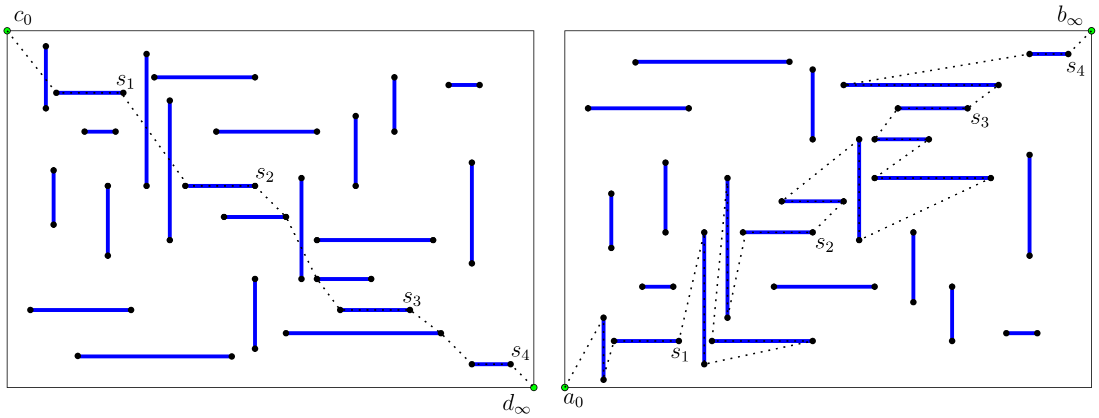



Figure 7: The initial path $\gamma$ (left) and the resulting alternating path (right).

### 2.5 Computational Complexity

The bottleneck in our algorithm is the use of Dilworth's theorem. The currently known best algorithm for finding a chain or an anti-chain of size $\sqrt{n}$ in an $n$ element partially ordered set is based on a reduction [11] to the *maximum bipartite matching* problem. Given a partial order on $n$ elements and $m$ comparable pairs, the problem is reduced to finding a matching problem in a bipartite graph with $2n$ vertices and $m$ edges. Hopcroft and Karp [13] gave a *deterministic* $O(m\sqrt{n}) = O(n^{2.5})$ time algorithm for the maximum bipartite matching problem. (Benczúr et al. [3] proposed an $O(nh)$ time algorithm where $h$ is the number of *directly comparable pairs* $s_1 < s_2$ such that there is no $s$, $s_1 < s < s_2$. This does not improve on our bound, since in our problem $h$ may be $\Theta(n^2)$.) Recently, Mucha and Sankowski [16] have found a *randomized* algorithm of expected $O(n^{2.38})$ runtime for the maximum bipartite matching problem based on the Hopcroft-Bunch Gauss elimination algorithm [6], which is dominated by the runtime of matrix multiplication [7].

A simple shortest path homotopic to a given simple curve among $n$ obstacle points can be computed in $O(n \log^2 n)$ time by an algorithm of Bespamyatnikh [4]. We use $O(n)$ `Expand` operations because each operation attaches a new segment endpoint to $\gamma$, and no segment endpoint is ever detached from $\gamma$. All `Expand` operations can be completed in $O(n^2 \log^2 n)$ time.

## 3 Upper bound construction

**Lemma 7.** *For every $k \in \mathbb{N}$ there is a set of $n = \frac{4}{3}(4^k - 1)$ pairwise disjoint axis-parallel line segments in the plane such that the length of any alternating path is at most $4(2^k - 1) = \sqrt{12n + 16} - 4$.*

*Proof.* We describe a construction $S_k$ of $\frac{4}{3}(4^k - 1)$ disjoint axis-parallel line segments recursively for every $k \in \mathbb{N}$. $S_1$ is a set of four disjoint line segments placed along the four sides of a square. For $k > 1$, we obtain $S_k$ as follows. Consider a disk $D_k$ and a square $Q_k$ such that they have a common center of symmetry and both $D_k \setminus Q_k$ and $Q_k \setminus D_k$ are non-empty. $S_k$ consists of four chords of $D_k$ along the four sides of $Q_k$ and four copies of $S_{k-1}$ in the four components of $D_k \setminus Q_k$ (see Fig. 8). We call the four segments along sides of $Q_k$ the *principal segments* of $S_k$. By construction, $|S_1| = 4$

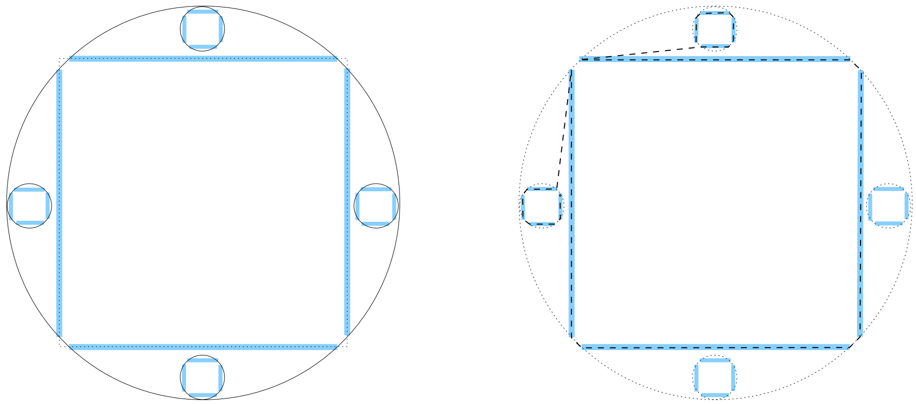


Figure 8: $S_2$ (left) and a longest alternating path for $S_2$ (right).

and $|S_k| = 4 + 4|S_{k-1}|$, so $|S_k| = 4 + 4^2 + \ldots + 4^k = \frac{4}{3}(4^k - 1)$. Note that the construction $S_k$ contains a total of $4^{\ell-1}$ copies of the construction $S_\ell$ for every $\ell$, $1 \le \ell \le k$.

It remains to be shown that the longest alternating path in $S_k$ passes through at most $4(2^k - 1)$ segments. We prove by induction on $\ell$ that an alternating path can contain a principal segment from at most $2^{k-\ell}$ copies of $S_\ell$, $1 \le \ell \le k$, within $S_k$. Since every copy has four principal segments, this totals to $4(1 + 2 + \ldots + 2^{k-1}) = 4(2^k - 1)$ segments.

For the ease of the induction argument, we actually prove a stronger statement. An alternating path $\alpha$ has at most $2^{k-\ell}$ (disjoint) maximal portions such that each portion passes through segments of only one copy of $S_\ell$, $1 \le \ell \le k$, in $S_k$. The statement clearly holds for $\ell = k$.

Assuming that the statement holds for all $\ell'$, $\ell < \ell' \le k$, we argue for $\ell$. Let $C$ be a copy of $S_{\ell+1}$ and let $\alpha_C$ be a maximal portion of $\alpha$ that passes through segments of $C$ only. If $\alpha_C$ contains segments from two different copies of $S_\ell$ in $C$, then $\alpha_C$ must pass through a principal segments of $C$. Therefore, if $\alpha_C$ contains segments from a copy of $S_\ell$, then at least one endpoint of the path $\alpha_C$ must be in that copy. Consequently, $\alpha_C$ has at most two maximal portions such that each of them contains segments exclusively from one copy of $S_\ell$ within $C$. □

For other integers $n$, $\frac{4}{3}(4^{k-1} - 1) < n < \frac{4}{3}(4^k - 1)$, we can give similar but unbalanced constructions. Let us assume that $n = 4 + m_1 + m_2 + m_3 + m_4$ such that $0 \le m_i \le \frac{4}{3}(4^{k-1} - 1)$ for $i = 1, 2, 3, 4$. Consider a disk $D_k$ and a square $Q_k$ such that they have a common center of symmetry and both $D \setminus Q$ and $Q \setminus D$ are non-empty. We place four segments along the chords of $D$ along the four sides of $Q$. Then in the four components of $D \setminus Q$, we place copies of construction with $m_i$, $i = 1, 2, 3, 4$, segments respectively. Applying Lemma 7 for each copy of size $m_i$, we conclude that no alternating path can be longer than $4 + 2\max_{1 \le i \le 4}(\sqrt{12m_i + 16} - 4) \le 2\sqrt{12n + 16} + 8$. This proves Theorem 1 (ii).

# 4 Protruded axis-parallel segments

In this section, we consider a protruded set $S$ of $n$ pairwise disjoint axis-parallel segments. Recall that a set $S$ of line segments is protruded if the extensions of the line segments have pairwise disjoint relative interiors. First we prove Theorem 2 and give an $O(n \log n)$ time algorithm that constructs an alternating path passing through all segments of $S$; then we prove Theorem 3.

Let $B$ be the axis-parallel bounding box of $S$. We denote by $a_0$ and $b_\infty$ the lower left and the upper right corner of $B$, respectively. We compute the (unique) convex partitioning of the free space $B \setminus \bigcup S$ in $O(n \log n)$ time. For every segment endpoint $v$, we extend the incident input segment beyond $v$ until it hits another segment or the boundary of $B$. This can be done by four sweep-line algorithms, one for each axis-parallel direction. Thus we partition $B$ into $n+1$ rectangular faces. Consider a face $F$ of the partition. A corner $v_F$ of $F$ can be one of the four corners of $B$; otherwise $v_F$ is a point where the extension of a segment $ab \in S$ beyond one of its endpoints, say $a$, hits another segment or the boundary of $B$. In this case, we say that $v_F$ *corresponds* to the segment endpoint $a$. The segment endpoint corresponding to a corner $v_F$ of $F$ lies on the boundary of $F$, because $S$ is protruded. We say that the union $G$ of some of the faces of this convex partition is a *staircase polygon* if $G = LL(G)$.

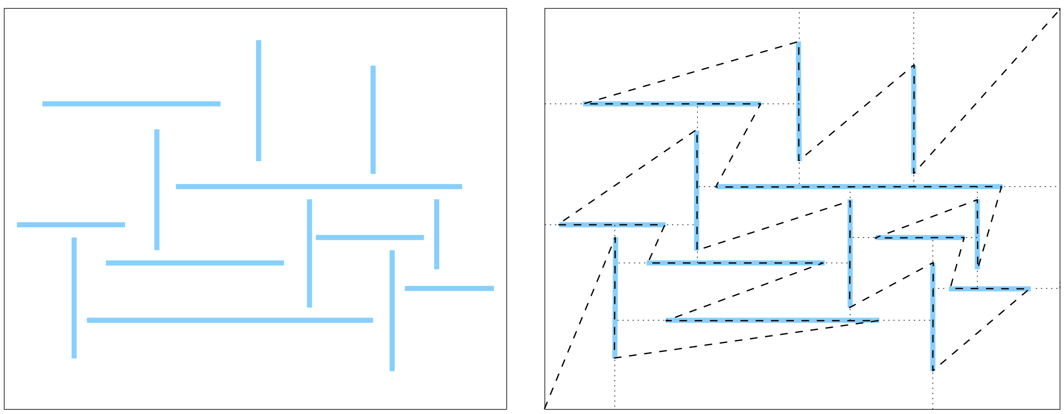


Figure 9: A protruded set of 14 segments (left) and our alternating path $\alpha$ (right).

**Lemma 8.** *Consider a protruded set $S$ of pairwise disjoint axis-parallel segments in a bounding box $B$, and a staircase polygon $G$.*

*(i) If the length of $\partial LL(G)$ is $k$, then it contains $k+1$ segment endpoints of $S$.*

*(ii) If there are $\ell$, $0 \le \ell < k$, segments of $S$ such that they intersect $\partial LL(G)$ but their endpoints are not in $\partial LL(G)$, then at least $\ell + 1$ segments of $S$ are contained in $\partial LL(G)$.*

*Proof.* A polygonal path of length $k$ has $k+1$ vertices. Since $S$ is protruded, at every vertex of $\partial LL(G)$ the extension of a segment hits the relative interior of another segment or the bounding box $B$. The segment endpoint corresponding to every vertex of $\partial LL(G)$ lies on $\partial LL(G)$. This proves (i). For (ii), notice that $\partial LL(G)$ is a monotone decreasing curve, and so it intersects exactly $k$ segments of $S$. The intersection $\partial LL(G) \cap s$ is connected for every $s \in S$. Since $\partial LL(G)$ contains $k+1$ segment endpoints of $k - \ell$ segments, at least $\ell + 1$ of them are fully contained in $\partial LL(G)$. ☐

We are now ready to present a recursive algorithm that draws a simple alternating path $\alpha$ through all segments of $S$ (see Fig. 9). Initially, we set $i = 0$ and let $\alpha_0$ be the trivial (one-point) curve $\alpha_0 = \{a_0\}$. For every $i$, $i = 0, 1, 2, \ldots$, let $F_i$ be the face whose lower left corner corresponds to $a_i$. If $F_i$ is not the upper right face of the partition then let $b_{i+1}$ denote the segment endpoint corresponding to the upper right corner of $F_i$, where $a_{i+1}b_{i+1} \in S$. We append to $\alpha_i$ two new edges $\alpha_{i+1} := \alpha_i \cup a_i b_{i+1} \cup b_{i+1} a_{i+1}$ and put $i := i + 1$. If $F_i$ is the upper right face of the partition then we return $\alpha := \alpha_i \cup a_i b_\infty$ and terminate the algorithm.

Observe that the above recursive algorithm is correct. If $b_{i+1}$ corresponds to the upper right corner of a face, then $b_{i+1}$ is an upper endpoint of a vertical segment or the right endpoint of a horizontal segment. Therefore the other endpoint, $a_{i+1}$, of the segment $a_{i+1}b_{i+1}$ corresponds to the lower left corner of a face $F_{i+1}$. This ensures that our algorithm terminates only if $F_i$ is incident to the upper right corner $b_\infty$ of $B$, which does not correspond to any segment endpoint.

*of Theorem 2.* We show that the path $\alpha$ constructed by the above recursion is a simple Hamiltonian alternating path. It is clear that $\alpha$ is alternating by construction. It is partially monotone w.r.t. $S$ and, by Lemma 3, it is a simple curve. It does not cross any input segment because its visibility edges traverse faces of the convex partition. In order to prove that $\alpha$ passes through all $n$ segments, it is enough to show that $\alpha$ traverses all $n + 1$ faces of the partition, since every segment has an endpoint that corresponds to the lower left corner of a face. Let $G_i$ denote the union of faces that $\alpha_i$ traverses for $i = 1, 2, \ldots$. We show the following two Claims by induction in $i$.

(1) $LL(G_i) = G_i$, (that is, $G_i$ is a *staircase polygon*)

(2) At most one segment of $S$, namely $a_i b_i$, is contained in $\partial LL(G_i)$.

It follows immediately from Claim (1) that when the algorithm terminates and $b_\infty \in G_i$, then $G_i = LL(b_\infty) = B$ and so $\alpha$ has traversed all the faces of the partition.

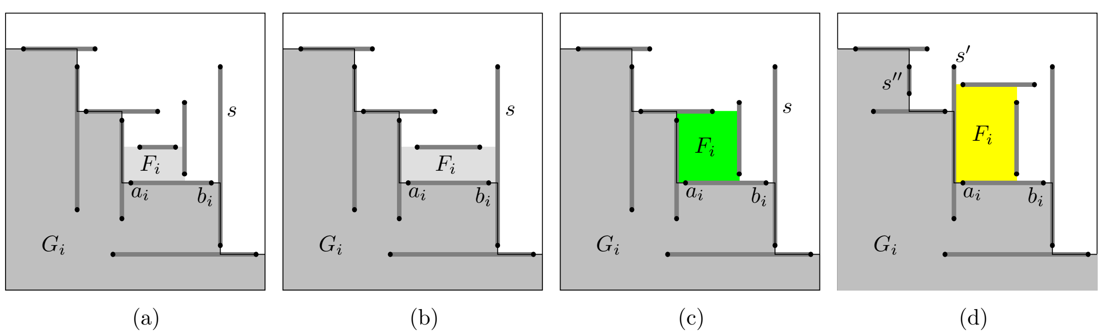



Figure 10: A staircase polygon $G_i$ and all segments of $S$ that intersect $\partial LL(G_i)$ or $\partial F_i$. Three possible positions of $F_i$ (a-c). An example where Claim (1) holds for $G_i$ but (2) does not (d).

In the base case $i = 1$, we have $G_1 = F_1$. Since $F_1$ is an axis-aligned rectangle incident to the lower left corner of $B$, we have $LL(G_1) = G_1$. There are two segments of $S$ along the boundary of $F_1$. The upper right corner of $F_1$ corresponds to an endpoint of one of these segments, which is contained in the boundary of $F_1$, and so the other segment is not contained in the boundary of $F_1$.

Suppose that Claims (1) and (2) hold for $i$, $1 \le i < n$. To prove Claim (1) for $G_{i+1}$, it is enough to show that the face below the lower right corner of $F_i$ and the face left of the upper left corner of $F_i$ are in $G_i$. Assume w.l.o.g. that $a_i b_i$ is horizontal. If we denote by $w(b_i)$ the right endpoint of the extension of $a_i b_i$, then $w(b_i)$ lies on the boundary of $F_{i-1} \subseteq G_i$ and it is in the relative interior of a vertical segment $s \in S$, where Claim (2) implies that $s \not\subset \partial LL(G_i)$. The face below the lower right corner of $F_i$ belongs to $G_i$ because this corner must be to the left of $s$ (see Fig. 10 (a-c)). Next, suppose that the cell to the left of the the upper left corner of $F_i$ does not belong to $G_i$ (see Fig. 10 (d)). This implies that there is a vertical segment $s' \in S$ along the left side of $F_i$, whose upper endpoint does not correspond to a corner of any face of $G_i$. It follows from Lemma 8(ii) that there must be segment $s'' \in S$, $s'' \neq a_i b_i$, fully contained in $\partial LL(G_i)$, contradicting the induction hypothesis. This proves $LL(G_{i+1}) = G_{i+1}$.

To prove Claim (2) for $G_{i+1}$, we can already use Claim (1). We start with showing that $a_{i+1}b_{i+1}$ is contained in $\partial LL(G_{i+1})$. Since $G_{i+1} = LL(G_{i+1})$, we have $b_{i+1} \in \partial LL(G_{i+1})$. If $a_{i+1}b_{i+1}$ is horizontal (vertical), then the face above (to the left of) $a_{i+1}$ is not in $G_{i+1}$, and $G_{i+1} = LL(G_{i+1})$ implies that no face above (to the left of) $a_{i+1}b_{i+1}$ is in $G_{i+1}$. Finally, we show that no segment $s \in S$, $s \neq a_{i+1}b_{i+1}$, is contained in $\partial LL(G_{i+1})$. By induction, $\partial LL(G_i)$ contains the single segment $a_i b_i$. We can obtain $\partial LL(G_{i+1})$ from $\partial LL(G_i)$ by replacing some portions of it by the upper and right sides of $F_i$. Therefore, only the two segments along the upper and the right side of $F_i$ may be contained in $\partial LL(G_{i+1})$. Since $S$ is protruded, the upper right corner of $F_i$ lies in the relative interior of one of these two segments, and so at most one of them (i.e., $a_{i+1}b_{i+1}$) may be contained in $\partial LL(G_{i+1})$.

□

Based on the proof of Theorem 2, we can obtain an 1-2-alternating path passing through all segments of a set of pairwise disjoint axis-parallel segments.

*of Theorem 3.* Assume that we are given a set of $n$ pairwise disjoint axis-parallel line segments in the plane. We can protrude the set of segments as follows.

1. Choose a sufficiently small $\varepsilon > 0$,
2. extend every vertical segment beyond its upper endpoint until the new endpoint is at $\varepsilon$ distance from another segment or the boundary of $B$,
3. extend similarly all horizontal segments beyond their right endpoints,
4. extend similarly all horizontal segments beyond their left endpoints,
5. extend similarly all vertical segments beyond they lower endpoints.

This process requires four separate sweep-line algorithms and can be completed in $O(n \log n)$ time. If $\varepsilon > 0$ is sufficiently small, then the resulting set of segments is protruded. By Theorem 2, there is a simple Hamiltonian alternating path passing through all protruded segments.

Consider the (unique) convex partition of the protruded set of segments. Even though the segment endpoints of the protruded segments corresponding to the corners of a cell $F$ lie on the boundary of $F$, this is not always true for the original segment endpoints of $S$. We show that for every cell $F$, the segment endpoint $v_F$ corresponding to the *lower left corner* of $F$ or the segment endpoint $w_F$ corresponding to the *upper right corner* of $F$ lies on the boundary of $F$.

First let us suppose that $w_F = b$ for a horizontal segment $ab \in S$. If $b$ is not on the boundary of $F$, then the upper extension of a vertical segment $s' \in S$ hits the segment $bw_F$. This is impossible, because the upper extension of the vertical segment $s'$ was drawn prior to the right extension of $ab$. Next, suppose that $w_F = b$ for a vertical segment $ab \in S$. If $b$ is not on the boundary of $F$, then the right extension of a horizontal segment $s' = a'b' \in S$ lying along the lower side of $F$ hits the segment $bw_F$. If $v_F = a'$, then $a'$ lies on the boundary of $F$. If $v_F = a''$ for the vertical segment $a''b'' \in S$ along the left side of $F$, then $a''$ lies on the boundary of $F$, otherwise the left extension of a horizontal segment along the upper side of $F$ would hit $a''v_F$, which is impossible since the lower extension $a''w_F$ was drawn after all horizontal extensions.

Let $v_F$ and $w_F$ be the segment endpoints corresponding to the lower left and the upper right corners of $F$, respectively. Since at least one of $v_F$ and $w_F$ lies in the union of on the boundary of $F$, we can connect $v_F$ to $w_F$ via a polygonal path of length two that lies in the union of the interior of $F$ and the segment extensions beyond $v_F$ or $w_F$. These polygonal paths are pairwise disjoint for every face $F$. The input segments $S$ together with the paths between $v_F$ and $w_F$ form a 1-2-alternating path $\beta$. By Theorem 2, the protruded set of segments admit a Hamiltonian alternating path $\alpha$. Since all segments of $S$ along $\alpha$ are part of the path $\beta$, we conclude that $\beta$ passes through all segments of $S$ and so it is a simple Hamiltonian 1-2-alternating path. □

## 5 Concluding remarks

We have shown that the longest alternating path in a set of $n$ disjoint axis-parallel line segments in the plane passes through $\Omega(\sqrt{n})$ segments and this bound is best possible in the worst case. Our proof is based on Dilworth's theorem and computation of convex arcs among polygonal obstacles. We close the paper with a few open questions.

1. What is the computational complexity of finding the longest alternating path for an input of $n$ pairwise disjoint axis-parallel (or generic) line segments?
2. What is the maximal $f_k(n)$ such that every set of $n$ pairwise disjoint segments with $k$ distinct orientations, for a fixed $k \in \mathbb{N}$, admits an alternating path of length $f_k(n)$? Our upper bound construction readily generalizes and gives a bound of $O(n^{1/(1+\log k)})$, but our lower bound algorithm works only for two distinct orientations.
3. Is there an algorithm for finding a chain or anti-chain of size $\Omega(\sqrt{n})$ in a poset of size $n$ (or, at least in the poset we defined on horizontal segments) that beats the complexity bounds presented in Subsection 2.5, dominated by bipartite matching algorithms ?
4. Is there a 1-2-alternating path through all segments of any protruded set of pairwise disjoint segments (i.e., without any restriction on the number of distinct orientations)?

## Acknowledgement

I would like to thank the anonymous referee for numerous useful comments and suggestions.

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